# The (Simplified) Story of Quantum Mechanics, Version 1

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## Contents

# 1 Brief Summary and Useful Info

These notes aim to give you a quick introduction to quantum mechanics. We begin with a motivating introduction, then go on to talk about some history of the field and strange experimental observations. I then develop the quantum formalism a little and introduce some ideas about measurement, the uncertainty principle, differential equations, wave-functions, and the Schrodinger equation. I then cover the fun example of quantum tunneling and some of its applications.

These notes are intended to give you an introduction that goes deeper than the wishy-washy pop-sci articles or quirky video explainers you have seen before. There will be scary scary math showing up from time to time, but I will do my best to give you a conceptual understanding of what the equations are saying. We ought not be afraid of equations, for they are simply a very concise and powerful language for describing the phenomena we observe and model in physics, and they are a core part of understanding the concepts at hand.

Hopefully these notes are both readable and engaging for the average high-school senior. Certainly it will not be easy, and certain sections may require you to go search up more information, re-read other parts, or go learn on your own. Quantum mechanics often gets a bad rep for being challenging and difficult, but it is really the novelty that makes it hard. Think of it like learning a new language: there are a whole class of new phenomena and new formalisms that you have not encountered before, and it will take time to get used to things. Don't be intimidated, and don't feel disheartened and if you don't understand things too well the first time through. Let your confusion and lack of understanding be motivation to continue through this mysterious land of quantum physics so that you may one day come out with a solid grasp of the concepts at hand!

Throughout these notes, you may see some sections have a bunch of asterisks next to them. These asterisks correspond to an experiment related to the subject being discussed or some explanation on how we might measure such a quantity in the lab. These little segways are meant to give you some physically grounding about the often nebulous and strange theory of quantum mechanics.

You may also see lots of typos, grammatical errors, or even math mistakes. These notes were typed up rather hastily, so do your best to ignore them. Sorry...

# 2 Why Quantum Mechanics

I will give a brief introduction to quantum mechanics, and before doing so, it is perhaps wise to briefly talk about classical mechanics. Classical mechanics is the usual way in which we perceive the world - forces, balls bouncing against the ground, skates gliding on ice. However, as we approach atomic scales, classical mechanics begin to break down, and we need a new physical theory to model the behavior of our system. This is where quantum mechanics arises. The name "quantum" arises from the quantized nature of the physics at this scale. Quantities such as energy, momentum, angular momentum, a strange thing called spin, and other such characteristics take on discrete values, as opposed to the continuous quantities we are typically familiar with. At our human size, we can run at any pace we like, there is no discrete step-size we must accelerate at or some forced increment of speed we can move at. However, things like atomic particles have specific allowed momentums and energies that they can be at. This is the quantized nature of physics.

Note that classical and quantum mechanics are not necessarily in direct opposition to one another, nor is one "better" than the other. They are just situational theories that ought to be applied to the correction system. You would be quite upset if you had to calculate the strain a bridge could take, or how quickly it takes for a car to accelerate to sixty miles per hour and were only equipped with quantum mechanics. You would similarly be ill-equipped to study the dynamics of electrons and protons orbiting a nucleus with classical mechanics.

However, despite my seemingly hard-lined separation of the two theories in the previous paragraph, it should be noted that you could actually model the dynamics of electrons and protons in an atom with classical mechanics, you would just be a little bit off. A key idea is that classical and quantum mechanics are actually very well-connected: As the scale of your system becomes larger, quantum theory predicts the usual elements of classical elements, and similarly, you can study quantum systems quite well using classical mechanics, though what you will get back will typically be approximations or predictions that only apply in certain limits.

It should be noted that oftentimes in these notes, I will simplify concepts and hide certain details to make things clear and to not scare you, the reader, too much. Sometimes, I will present things that are straight up wrong for the sake of making a clear point or to explain something in an easier way. Usually, I will try to warn you that that is what I'm doing, but sometimes I won't tell you, and you will be blissfully unaware. Know that I am hiding some of the dirty secrets away from you for your own good. However, do take these notes with a grain of salt. If you want to be a smartass and show off to your high-school aged peers about your knowledge of quantum mechanics, you can tell them about what you have learnt from these materials with confidence, but trying to show up anyone who has taken a basic college course in quantum mechanics might end with you getting caught with your pants down.

# 3 Discrepancies

I'll present some theoretical and experimental holes that arise if only classical mechanics existed and discuss why quantum theory patches up those holes. We will see some effects that are totally baffling and undescribable through a classical framework, and hopefully this provides some apt motivation as to why the theory of quantum mechanics began to be developed.

## 3.1 The Ultraviolet Catastrophe

In physics there are objects known as blackbodies that completely absorb any light or other types of electromagnetic waves that run into them. Of course, these are only theoretical objects, as no physical thing in the real world can be entirely perfect, but nevertheless they are useful to study.

When a blackbody absorbs electromagnetic waves, they will then eventually emit this energy in the form of radiation once it has reached thermal equilibrium with its environment. To disentangle this phrase a bit more, we can think of shooting a bunch of electromagnetic waves at a blackbody that's sitting in a room. The blackbody will absorb these waves and get excited and perhaps heat up very suddenly, and its temperature will rise higher than that of the room. However, eventually the blackbody will relax or maybe the room temperature will increase until things settle and the temperature of both room and blackbody no longer change. After this settling has occurred, blackbody radiation is emitted.

Prior to the development of quantum mechanics, the intensity of the radiation for a specific wavelength $\lambda$ emitted by a blackbody was described by the Rayleigh-Jeans law of form

$$I_\lambda(T) = \frac{2ck_BT}{\lambda^4}. \tag{1}$$

There are two constants in this equation $c$ and $k_B$ which we won't worry about. Let's instead examine the two variables $T$ and $\lambda$, the temperature and the wavelength respectively. As $T$ increases, the radiated intensity increases, which intuitively makes sense as higher temperatures correspond to higher energy, and yield more intense radiation. We also notice $\lambda^4$ in the numerator, leading us to think that there is a very heavy dependence on the wavelength, as the 4th power implies that a slight change in wavelength will have a large effect on the intensity. As $\lambda$ decreases, the intensity of radiation decreases, which also makes sense, as lower wavelengths are more energetic (Think about all of the talk about blue light filters and avoiding blue light at night! Blue light is on the lower end of the visible spectrum and thus carries more energy, meaning it is harsher on the eyes.).

You have have noticed one slight issue with this equation. As the value of $\lambda$ gets smaller and smaller and smaller, the radiated intensity becomes huge and eventually explodes to infinity. If you are unfamiliar with this concept, let's examine some values. For convenience, let's imagine that the numerator of equation 1 happens to be 1 so that we can examine the denominator more conveniently (you might think this is slightly arbitrary, but we do this just to get a qualitative feel of how the intensity will behave for ever smaller wavelengths). Performing these simplifications to better understand the general behavior of functions is something that is quite commonly done in physics!

We'll examine a couple of different wavelengths. Wavelengths are sometimes described by micrometers ($\mu m$) or $10^{-6}$ meters.

$$
\begin{aligned}
\lambda = 3\mu m : \qquad & I_\lambda = \frac{1}{(3\times 10^{-6})^4} && = 1.23\times 10^{22} \\
\lambda = 0.78\mu m : \qquad & I_\lambda = \frac{1}{(0.78\times 10^{-6})^4} && = 2.70\times 10^{24} \\
\lambda = 0.1\mu m : \qquad & I_\lambda = \frac{1}{(0.1\times 10^{-6})^4} && = 1.00\times 10^{28} \\
\lambda = 0.01\mu m : \qquad & I_\lambda = \frac{1}{(0.01\times 10^{-6})^4} && = 1.23\times 10^{32} \\
\lambda = 0.001\mu m : \qquad & I_\lambda = \frac{1}{(0.001\times 10^{-6})^4} && = 1.23\times 10^{36}
\end{aligned}
$$

and as we go smaller and smaller, this value will only continue to grow and grow. The theoretically predicted values were completely different from the experimental results, and something was deeply, deeply wrong (hence the name "catastrophe).

The remedy to this catastrophe was provided by Max Planck, and he came to this solution in quite a weird way. He postulated that electromagnetic radiation (like the kind that is emitted by blackbodies) can only be emitted or absorbed in discrete chunks, which he termed quanta. To be explicit, a single quanta of electromagnetic radiation for a given wavelength $\lambda$ carried energy

$$E = \frac{\hbar c}{2\pi\lambda} \tag{2}$$

where $c = 3\times 10^8$m/s is the constant that describes the speed of light and $\hbar = 6.67\times 10^{-34}$J/s is the reduced Planck's constant (pronounced H-bar). Now, you may be wondering, this seems to be a bit out of nowhere, but Planck did have his reasons as well as a large quantity of experimental results to try and match his theory to. This is also how he deduced the value for $\hbar$, which at first glance seems a rather strange and arbitrary value. However, experimental measurements of this constant show that it is quite accurate, and Planck's constant is now considered a *fundamental constant* in physics, (meaning they govern the most basic and important principles of our physical world). I won't really delve into all the reasons, but the important thing is that with the condition that energy is discrete or quantized, the Rayleigh-Jeans law is then modified to be

$$I(\lambda, T) = \frac{\hbar c^2}{2\pi\lambda^5}\frac{1}{\exp\left(\frac{\hbar c}{2\pi\lambda k_B T}\right) - 1} \tag{3}$$

where $c$ is the speed of light, $k_B$ is a constant called the Boltzmann constant, $T$ is the temperature, $\lambda$ is the wavelength, and $\hbar$ is the reduced Planck's constant. The "exp" simply means the exponential and we use that form sometimes when there are a lot of variables in the exponent (for example, $e^x$ is the same as $\exp(x)$).

Now this equation looks a little bit scarier than before, but like we did last time, we shall just play around with it and qualitatively understand what's going on. Since this equation is a bit more crowded, let's break things down even further:

1. **Increase $T$:** As $T$ increases, the value inside the exponential gets smaller. This means the exponential itself rapidly get smaller and approach 1, causing the denominator in the expression $\frac{1}{\exp\left(\frac{\hbar c}{2\pi\lambda k_B T}\right)-1}$ to decrease. This thus increases the overall value of the intensity. As we would expect, when temperature increases, intensity increases.

2. **Decrease $\lambda$:** If we decrease the wavelength $\lambda$, the most striking effect we first notice is that the denominator in $\frac{\hbar c^2}{2\pi\lambda^5}$ rapidly increases, which at first glance causes the intensity to once again explode. However, when we examine the exponential, we see that as $\lambda$ decreases, the exponential $\exp\left(\frac{\hbar c}{2\pi\lambda k_B T}\right)$ gets very large, as $\lambda$ is in the denominator. As the exponential gets bigger, the denominator in the term $\frac{1}{\exp\left(\frac{\hbar c}{2\pi\lambda k_B T}\right)-1}$ gets very large, causing the term to become very small. Thus we see Planck's modification to the Rayleigh-Jeans equation prevents huge values in intensity for small wavelengths as the term $\frac{\hbar c^2}{2\pi\lambda^5}$ and the term $\frac{1}{\exp\left(\frac{\hbar c}{2\pi\lambda k_B T}\right)-1}$ effectively work in opposition to one another when wavelength is varied.

To make things more clear, we can compare the results between the classical and the quantum results (aka the Rayleigh-Jeans equation and Planck's modification) shown in figure 1:

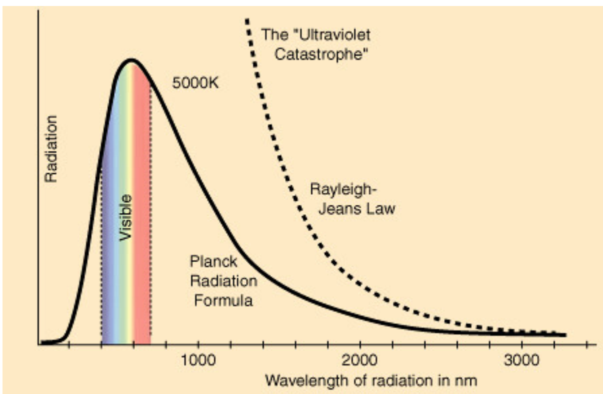


Figure 1: Qualitative plot of the radiated intensity by a blackbody at a fixed temperature of 5000 Kelvin. The dashed line is the prediction by the Rayleigh-Jeans law and the dashed line is Planck's modification. The location of the visibile spectrum is colored in on the $x$ axis.

From the figure, we see that at smaller wavelengths (such as those in the visible spectrum), the Rayleigh-Jeans law makes entirely unphysical predictions while Planck's modification gives us the correct results that match those that are found through experiment.

Thus we see that with the quantization of energy to discrete values of $\frac{\hbar c}{2\pi\lambda}$, Planck was able to mathematically resolve the ultraviolet catastrophe. Let us examine the physical consequences of this discovery. Instead of electromagnetic radiation being a continuous range of energies, they are instead discretized packets that come in certain values. This also means the wavelengths of emitted light are also quantized, as energy is described to be $E = \frac{\hbar c}{2\pi\lambda}$. From this equation, we can also infer that for certain wavelengths of radiation, there can only be certain values of energies that are allowed. Higher energies can essentially only be attained by stacking up smaller pieces of quanta, much like the ladder denoted in 2.

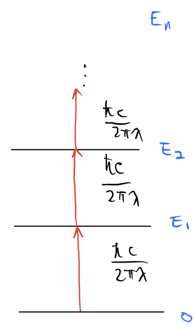


Figure 2: A schematic representation of how larger discrete energy values can be reached by "laddering" up single bits of quanta. In this instance, we climb up the energy ladder with quanta $E = \frac{\hbar c}{2\pi\lambda}$.

You may also find it interesting that from figure 1 we can see that at larger scales, the classical Rayleigh-Jeans theory and the quantum Planck theory actually match up fairly well. It is only as we go to smaller and smaller length scales that the two diverge and the classical theory is no longer able to properly describe the physics as we talked about in 2! So if you are working with wavelength about $3000nm$ ($3 \times 10^{-6}$ meters), the classical theory is perfectly fine to use. Since the graph tails off a bit towards the end, I will plot both classical and quantum predictions for larger wavelengths so that you are more convinced: We see that the curves are pretty much exactly the same!

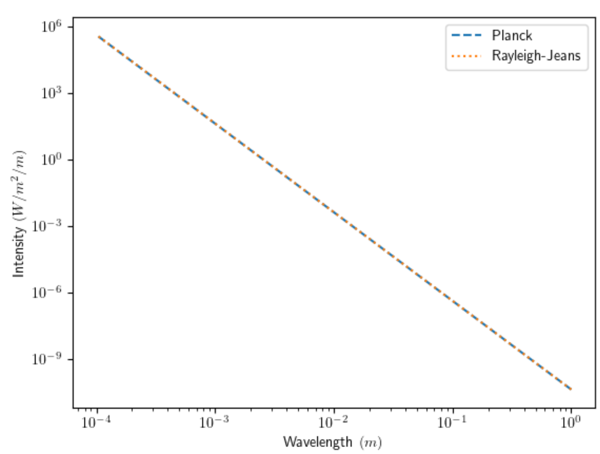


Figure 3: Plot of the Rayleigh-Jeans and Planck modification for blackbody radiation from $3000nm$ to $1m$. The plot is in log scale for clarity.

However, after all this, you might still be upset by the rather handwavey reasoning of

"Planck tried a weird idea and did some math. It happened to work out. Therefore we need quantum mechanics." Unfortunately, for now that is the best I can do. Planck's intuition to perform this quantization was (I think) due to a need to match experimental results and also a clever bit of dimensional analysis and it was the jumping off point for the development of quantum theory. For now, things are not too satisfying, but as we go on and build up more, things will hopefully be a bit more reasonable.

## 3.2 Photoelectric Effect

Schematically, the photoelectric effect works as follows: some light hits a material (or as physicists like to say, is incident upon a material) and as a result, electrons are emitted from the material. A cartoon displaying this effect is shown in figure 4.

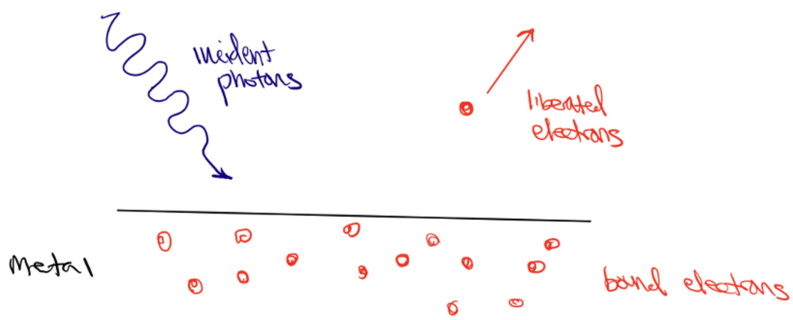


Figure 4: Cartoon of the photoelectric effect. Incident photons (blue squiggle) liberate electrons (red dots).

Experimentally, we observe that regardless of the intensity of the incident light, it must have a certain threshold frequency in order to successfully dislodge an electron and cause its emission from the material. No matter how powerful the beam of light is, if it is not above a certain frequency, no electrons are emitted! This is in conflict with the classical theory, which characterizes light to be a wave and reasons that a higher intensity wave ought to dislodge more electrons. To understand this more, let us look at the mathematics behind intensity and frequency. Intensity is defined to be

$$I = \frac{1}{2} c\epsilon_0 |\mathcal{E}|^2 \tag{4}$$

where $c$ and $\epsilon_0$ are constants we won't worry about, and $|\mathcal{E}|$ is the amplitude of the electric field. The electric field amplitude is the strength of the electric field, which in this case is the incident light (remember, classically, light can be treated as an electromagnetic wave). The amplitude of a wave is shown in figure 5.

Naively, we would expect a more "powerful" and higher intensity wave to have a higher probability of dislodging electrons. Cartoonishly, we can imagine the material to be a

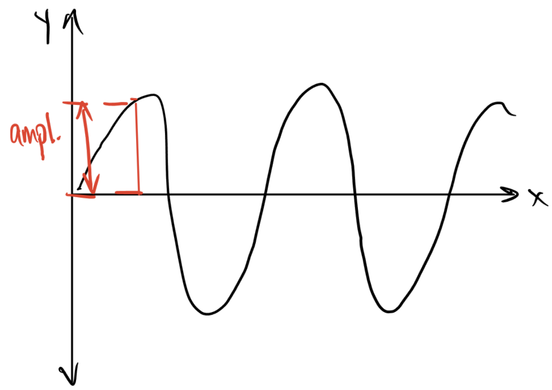


Figure 5: Amplitude of a wave.

tangle of vines holding clusters of grapes (the electrons). If we shake the vine harder (increase the amplitude of the electric field), shouldn't more grapes (electrons) fall off (be emitted)? Indeed, that is what the classical theory predicts. However, instead, we see an almost instantaneous frequency dependence. Below a certain frequency, no electrons are emitted. However, once we tune the frequency just high enough, electrons instantly start to come off of the material.

Let's think back to what we just learnt from the ultraviolet catastrophe. Before we do so however, I will make a rather suggestive substitution and relate the frequency $f$ and wavelength $\lambda$ of light by the formula

$$f = \frac{c}{\lambda} \tag{5}$$

where $c$ is the speed of light, a constant. In the above formula, we observe an inverse relationship. Since $\lambda$ is in the denominator of the fraction, as $\lambda$ smaller, $f$ must get bigger, and if $\lambda$ gets bigger, $f$ must get smaller. This equation makes a lot of physically sense. Frequency can basically be thought of as the number of cycles the wave makes in a single unit of time, while wavelength is, as the name suggests, how long the wave is. If the wave is very long, then it can't make as many cycles in a single period of time. Imagine you are whirling your arms around in circles for one minute. If you bend your arms, and thus shorten them (decreasing the wavelength), you can make more circles in that single minute than if you had your arms straight (longer wavelength). Your straightened and thus longer arms have to travel a greater distance than your bent arms, and thus a single cycle takes longer.

Ok, we went off on a bit of a tangent there but hopefully you are convinced that this relationship is true. Now, to sort out the ultraviolet catastrophe, Planck postulated that the energy of emitted electromagnetic radiation from a blackbody came in quantized chunks of form

$$E = \frac{\hbar c}{2\pi\lambda}. \tag{6}$$

If we apply equation 5 and some algebra, we can rewrite this to be

$$E = \frac{\hbar f}{2\pi}. \tag{7}$$

Here we see something interesting. If we apply Planck's quantization, we see that incident light (when treated as discrete chunks) of higher frequency is actually more energetic. In fact, the only thing that influences the energy of this quantized light is the frequency! The intensity and the field amplitude $\mathcal{E}$ don't matter at all! Indeed, this theory matches experimental results. What are the implications of this?

Developing a physical picture required Albert Einstein (who was actually awarded the Nobel Prize for his work on the photoelectric effect). Einstein proposed that light is actually made up of a bunch of single particles called photons, each carrying a discrete packet of energy. These photons bounce onto a material, and if they are carrying enough energy, dislodge an electron from the forces that hold them in the material. Whereas Planck opted to quantize the energy released from electromagnetic radiation to resolve some weird math, Einstein fitted a physical theory to experimental observations, moving quantum mechanics further forwards. Once again, we had a phenomena that was indescribable by purely classical theory that was now successfully fitted by quantum theory.

Notably, this energy quantization wasn't just a math trick; it had some legitimate grounding in how our world actually works. Next time you look up at the sun or at a lightbulb, you can imagine tiny little particles bouncing out one by one each carrying packets of energy $\frac{\hbar f}{2\pi}$ and entering your eyeballs. Photons!

## 3.3 Atomic Spectra

**(**Experiment: Measuring Atomic Spectra 7.1**)**

Now that we understand where photons come from and how they work, we can begin to examine another interesting phenomena that was experimentally rather confusing: atomic spectra.

You may be unfamiliar with what an atom's spectrum is, so let's first start with that. You may have learnt already that atoms have ground and excited states. The ground states are sort of like the "resting" state of the atoms and atoms are typically content with hanging out in these states. However, let's say we shoot a laser at a ground state atom. The atom will absorb some of the energy and jump up to an excited state. However, this excited state has some *finite lifetime*, meaning the atom cannot stay there forever (the reason for this is actually quite deep, but we will not get into it. You'll just have to trust me.) and eventually, to get back down to the comfortable ground state, the atom releases some energy in the form of a photon and falls back down to the ground state. Why is this photon released? Well, the atom has gone from a higher energy state to a lower energy state, and since energy must be conserved, the atom makes up the difference by way of this photon. The more energetic the laser is, the more excited the atom gets, and subsequently, when it decays back down, the more energetic the emitted photon is. That was a lot of words, so let's summarize:

1. Atom starts in ground state
2. We shoot a laser at the atom
3. Atom absorbs some amount of energy from the laser and climbs to the excited state
4. The atom isn't very stable in the excited state, and after some time, falls back to the ground state
5. Since the atom is going from a higher energy state to the lower energy state, it releases a photon to preserve conservation of energy

Let us take some random atom, such as Hydrogen. Now if we were naive classical theorists, we would expect the energy of the emitted photon to be any value that just depends on the properties of the laser we shoot it with. If we shoot 5 units of energy into the atom, it should emit a photon of 5 units when it decays from excited to ground state. If we shoot 6.2932992 units of energy, then the atom should emit a photon of that quantity when it decays down. If we shoot a very large beam with a wide mixture of different energies at the atom, then we would expect to see a spectrum that reflects all of those input energies. If we shot a wide beam considering all the wavelength of visible light at the atom, perhaps we would expect something that looks like figure 6. where the Hydrogen atom absorbs a wide range of all of the input wavelengths and

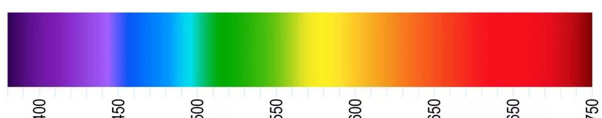


Figure 6: The naive classical physicists drawing of the atomic photon emission spectrum when we inject a beam of light containing wavelengths of 400-750. The numbers are in units of nanometers.

then proceeds to eject out photons with wavelengths that are the same as those inputs.

However, being the intelligent students we are, we have read the two previous sections and understand that this is a trap. When we examine the spectrum of Hydrogen, we actually observe the discrete spectrum shown in figure 7 where only a select couple of photons with discrete energies are ejected.

Now at this point, you have probably followed the story enough to realize that this is once again evidence of energy quantization. You may be a bit unfamiliar with how the numbers on the x-axis of the two spectra above are related to energy, and I have been rather vague about it and obscured many details for the sake of a schematic explanation, so let us delve into that a bit more. Figure 7 displays the wavelengths of

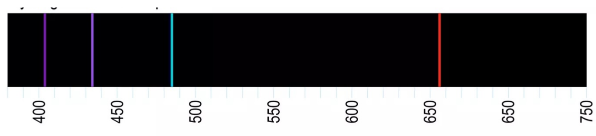


Figure 7: The actual spectrum of emitted photons from the atom when we shoot a beam of light containing wavelengths of 400-750. We see discrete lines at around 410nm, 440nm, 490nm, and 660nm. Once again, the numbers are in units of nanometers

photons emitted from the atom. Although these are wavelengths, if you recall equation 6, then you will see that these actually refer to energies, and so the plot is demonstrating energy quantization.

So, we see that only photons of certain energies are emitted. What does this mean about the atom? Let's try and break it down. If we apply the conservation of energy principle, we know that the emitted photon must contain the same amount of energy required to send the atom to its excited state. However, the atom only accepts photons of specific energies, which then means that the excited states of an atom are similarly quantized, and only accept photons of explicit energies! For hydrogen, if you shoot a laser beam of wavelength $\lambda = 550nm$, then nothing will happen, because hydrogen does not accept photons of such energy. If we shoot a beam of $600nm$ or $650nm$, similarly, nothing will happen. It is only when we shoot lasers of the explicitly allowed wavelengths, such as 410 or 440 nanometers will the hydrogen atom actually go to its excited state. In other words, the energy levels of atoms are also quantized, and there are only certain allowed energies that stimulate transitions to excited states.

This is actually a distinctive feature for different species of atoms. Atoms on different parts of the periodic table have different quantities of electrons and different atomic structures, and thus they have unique emission spectra. A couple of spectra for different atoms are shown in figure 8. Different atoms have radically different emission spectra!

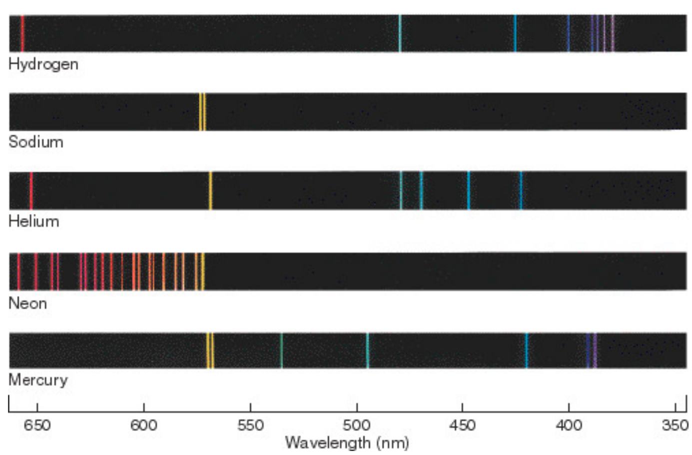


Figure 8: Atomic spectra for Hydrogen, Sodium, Helium, Neon, and Mercury. Image taken from Lifeng Astronomy web.

**Tangent: Some Useful Applications of Emission Spectra**

For pretty much all of the current atoms we have know of, we have a pretty good idea what their emission spectrum should look like. This is highly useful, as one application we have for emission spectra is to "fingerprint" and identify unknown substances through a method known as spectroscopy. Let's say we have a sample of gas that consists of a mixture of three different types of atoms that we have no knowledge of. If we shine a range of different light on the mixture and check to see what wavelengths are then emitted out, we will get a picture that looks like the atomic spectra of the three different atoms all superimposed on one another. By doing more experiments, we can figure out exactly which three atoms are present in the substance by matching our measurements to their individual known spectra. Pretty useful!

We can also test the purity of different substances this way: A company claims they are selling a drug that contains 50% Rubidium, 25%Cesium, and 25% Iron. This would be a pretty horrible and useless drug, but I am just using this as an example. If we perform the aforementioned spectra experiments, we would expect to see the spectral profile of Rubidium, Cesium, and Iron all super-imposed on one another. However, since there are 2 times the amount of Rubidium atoms as Cesium and Iron atoms, we should expect it the lines for Rubidium to be 2 times as bright! If that's not the case, then the company is either incompetent or they're scamming us.

# 4 Some Mathematics of Quantum Mechanics

## 4.1 Measurement

You may have heard from popular media that quantum mechanics is strange, unintuitive, and uncanny. Certainly the idea of quantization is an odd one to grasp and not quite how we would expect the world to behave. However, that is only one of the many strange phenomena that arise. Another equally fundamental and important one is the statistical nature of quantum mechanics.

What does that exactly mean? Before we get into it, let's consider a classical example, specifically for the case of measurement. Let's say I want to figure out how tall you are. That's pretty easy. I take a tape measure, stretch it from the top of your head to the tips of your toes and read out a number. With a single measurement, I find out that you are 5 feet tall. However, when we get into the quantum world, measurement is no longer as simple due to certain phenomena (The Heisenberg uncertainty principle) we will discuss later on.

In the quantum world, I take a measurement of your height. Strangely, my tape measure tells me you are actually 5 feet and 0.001 inches tall. Finding this to be a bit strange, I measure you again. However, my tape measure tells me you are 4 feet and 11.999 inches tall. Somehow, your height has fluctuated slightly between these measurements. I'm now quite concerned, and so I begin to measure you repeatedly. I find that you are 5.002 inches tall, then 4 feet and 11.998 inches tall, then 4 feet and 11.993 inches tall, then 5 feet and 0.002 inches tall. Maybe on one measurement, I find that you are exactly 5 feet tall. I keep measuring and measuring, maybe for 100 times, maybe for 1000 times. There is no consistency. The value I get always fluctuates slightly. Note that although I have used the "classical" example of measuring height, such fluctuations would only occur on the quantum scale (say for example, measuring the position of a proton). As a rather large and non-quantum human, I would always measure your height to be the same.

Instead, if I take all of the values I have measured and divide then by the amount of measurements I have made (meaning I take your average height), I find that your average height is 5 feet. It is common to call this value an *expectation value*, and we denote the expectation value of a variable with these brackets $\langle\rangle$. Mathematically, we can describe the expectation value of your height to be

$$\langle height \rangle = \sum_{i}^{n} \frac{m_i}{n} \tag{8}$$

where $m_i$ denotes each single measurement trial and $n$ denotes the total number of measurements. $i$ just denotes the the parameter that is changing, and allows us to distinguish between individual measurements. If this notation looks entirely foreign or you aren't too interested in the math mumbo-jumbo, feel free to skip this paragraph. The big symbol $\sum$ essentially means to sum over. In more plain English, what the equation is describing is that the expectation value of height is going to be a summation of all of the separate measurement trials $m_1, m_2, m_3, \ldots$ all the way up to $m_n$ divided by the total number of measurements $n$. If we want to be more explicit about it, we can write it this way:

$$\langle height \rangle = \sum_{i}^{n} \frac{m_i}{n} = \frac{m_1 + m_2 + m_3 + \cdots + m_n}{n}. \tag{9}$$

The $\sum$ symbol and the additional index $i$ just gives us a simple and compact way of writing this out.

When we talk about measurements in quantum, we talk about an average over many individual measurements. For instance, If I find that an atom has a momentum of 0.002, it means that when I measure this atom's momentum many many many times and take the average value, I find that it is is equal to 0.002. To understand quantum mechanics, you must become accustom to the notion that the measurements we make are actually statistical averages over many measurements.

### 4.1.1 Consequences: Vacuum Fluctuations

Now sure, you may be thinking "this is a bit strange, but its not too crazy. I guess when things are this small, they might fluctuate a little bit. Oh well." However, if we really think about it, we get some pretty weird phenomena. In my opinion, perhaps one of the strangest ideas that arises out of this statistical interpretation of things are fluctuations of the vacuum. Vacuum is typically thought of as something that is empty, in which no particles are present, and there is basically nothing there.

For the sake of example, let's think about a perfect river filled with water. Let's pretend this river is a quantum object and is subject to quantum phenomena. Let's say at quantum level 10, there is a lot of water in the river; at quantum level 5, there is an ok amount of water in the river; at quantum level 2, there is barely any; at quantum level 0, there is no water in the river. The river is a vacuum now, there is a complete absence of water in the river. However, what did we just learn? There is some statistical uncertainty to our measurement! We can't measure the river and consistently deduce that there is absolutely no water in the river. We might measure it once and find one single water droplet, measure it again, find nothing, measure it again and find a single strange anti-water droplet. Through repeated measurement, we will find that on average, there is absolutely no water in our river vacuum, but on a single trial basis, we will actually occasionally measure *something* instead of always *nothing*.

This class of phenomena are known as vacuum fluctuations, and are actually experimentally observed and simply a reality of our physical world. For example, we can actually observe fluctuations of the electromagnetic field even in the presence of total nothingness, in a manner similar to the plot shown in figure 9.

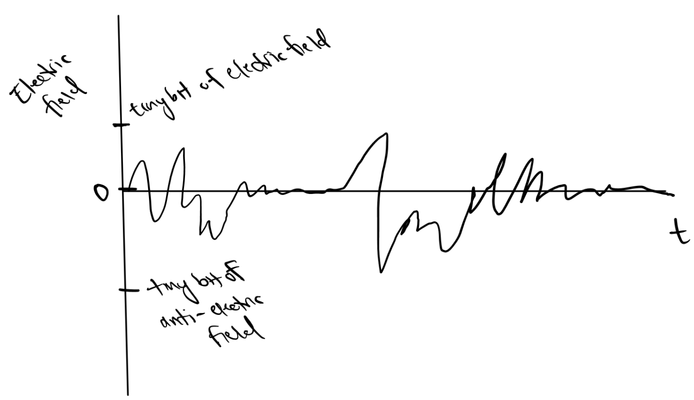


Figure 9: Fluctuations of the electromagnetic field in vacuum (there is nothing that should be generating such a field, but we are able to observe it anyways!). Note that these fluctuations average out to be 0, with the positive and negative measurements all canceling if we measure for long enough.

Let's focus on the vacuum fluctuations of the electromagnetic field for a quick second, cause I think its a very mind-boggling phenomena. So the electric field, when quantized, leads to photons, which you may know as the little particles that comprise of light (light is just an electromagnetic field!). In vacuum, when the electromagnetic field is supposed to be 0, there is still a possibility that very tiny numbers of photons or anti-photons can be produced. You may be wondering: this sounds totally made up! Show me some proof!

Well, if a particle were to be produced, then it ought to have an effect on the environment around it. Let's say I put two plates in vacuum and design them in such a way that they may interact with the vacuum electromagnetic fluctuations. With a really sensitive force sensor, I can actually measure the force exerted on the plates by the photons produced from the vacuum fluctuations! This is known as the Casimir effect.

We have some pretty interesting physical theories that do pretty well at modeling these effects, but they do require a good amount of mathematical machinery to understand, so if you want to truly understand what is going on in strange cases like this, then perhaps a career in physics may be of interest!

You may have the burning question of "Why is this the case?" In the next section, I will give some exposition on the mathematics that governs the "unreliability" of these results. However, I can't really tell you why quantum mechanics is inherently statistical. Nature just happens to be that way I guess...

## 4.2 Heisenberg Uncertainty

A natural question to ask is: if these measurements are fluctuating, is there some law governing exactly how much they fluctuate? Indeed there is: the uncertainty principle. I will first give a rather abstract description of this phenomena. Then, for the sake of simplicity, we will look at the mathematics of fluctuations in position and momentum, and then we will go on to discuss a more generalized case.

Let me first briefly give an abstract definition of the uncertainty principle and why we see the aforementioned fluctuations. The uncertainty principle dictates that there is a fundamental limit to how well we can measure the value of certain pairs of physical properties. To explain this, I will steal an analogy that is made in the Griffiths intro to quantum mechanics textbook that is not accurate but a good model to think things through with.

Let's say we have a long piece of rope. You shake the rope a bunch and begins to periodically bounce up and down, generating a wave. If someone asks you "where is the wave" you would have a hard time answering. Its a bunch of waves all spread out over the long rope. However, if someone asks you what the wavelength of the wave you just created is, you'd have a pretty easy time. All you'd have to do is measure the distance between two peaks and you'd have an answer. This situation is demonstrated in figure 10.

In contrast, if you gave the rope a sudden jerk, only a single peak would pop up and travel across the rope. Now, if someone asks you where the wave is, you can just look at the position where the peak is and measure that to get your answer. However, if they

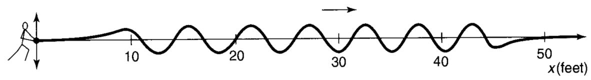


Figure 10: A rope with a well-defined wavelength and a poorly defined position.

ask you for the wavelength, you'd have a hard time, as there is only one peak and thus no way for you to really figure out the wavelength. This situation is shown in figure 11.

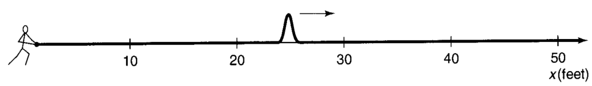


Figure 11: A rope with a well-defined position and a poorly defined wavelength.

Now, this example is extremely misleading. This isn't really similar to quantum uncertainty at all, it is more of a tricky situation. I wouldn't think too deeply about it or take it too seriously. However, it is a nice framework for thinking about how knowing more about one quantity can cause you to be uncertain about the other.

Let's move on to the quantum case. I mentioned earlier that there is a limit on how certain we are on the values of certain pairs of properties. For the sake of example, we will discuss position $x$ and momentum $p$. The famous uncertainty relation defining these two variables is given by

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{10}$$

where $\sigma_x$ is the standard deviation (describes how wide the distribution is) of the distribution describing the $x$ position, and $\sigma_p$ is the standard deviation of the distribution describing the $p$ momentum. A drawing of how this balance is kept is shown in figure 12.

This relationship effectively tells us that there will always exist some uncertainty in our measurements of position and momentum, and characterizes the "fuzziness", or uncertainty in our measurement. No matter how well we do, the product of the standard deviations of the distributions characterizing position and momentum will always be non-zero, and thus will never be totally precise. You can think about our measurements as being "smeared out".

Furthermore, we will always have to compromise. A more precise measurement of momentum will yield a less precise measurement of position, and a more precise measurement of position will yield a less precise measurement of momentum, as the two variables are always bound by the product of their standard deviations. This leads to some other fun tricks we can play, such as a phenomena known as "squeezing", which I will not explain and leave as a mystery for you to pursue (or get confused by).

Though we have explained this principle for position and momentum, one can derive a

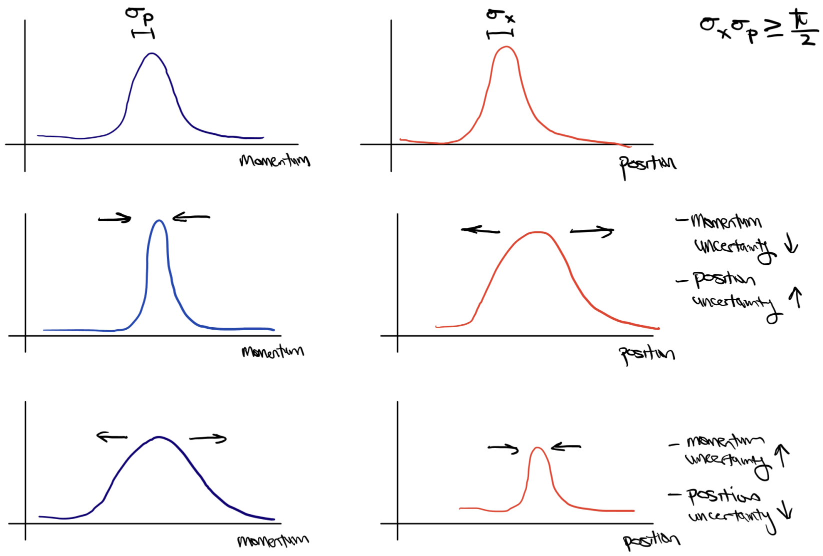


Figure 12: Trade-offs between measurement precision of position and momentum. Measuring position more precisely causes more uncertainty in momentum, and vice-versa.

general uncertainty principle of form

$$\sigma_A \sigma_B \geq \frac{1}{2}|\langle [A, B] \rangle| \tag{11}$$

where $A$ and $B$ are some pair of variables. I'm also not really going to explain this equation at all, I just want to show you that there is a general way to characterize different pairs.

The uncertainty principle is why we look at averages of many measurements in quantum mechanics! There is always some inherent uncertainty, so to get an actual answer, we have to take the average so that these fluctuations don't bias our results.

### 4.3 The Quantum Framework and Differential Equations

Let's briefly summarize some of the characteristics of quantum mechanical systems we have found out about:

1. Quantum mechanics is probabilistic - to make a proper measurement, we measure a quantity many many times and take the average
2. Quantum mechanics has some inherent uncertainty and we can only measure quantities so precisely
3. When things in your system get larger, quantum mechanics must eventually act the same as classical mechanics

With these characteristics in mind, we need to figure out some framework that can properly describe our quantum systems. At the end of the day, physics needs a mathematical description to model the dynamics of such systems. How does a bunch of quantum particles evolve in time? How certain quantum quantities change when we look at them in different places?

### 4.3.1 Classical Mechanics and Differential Equations

In classical mechanics, some familiar methods of modeling such things may be Newton's law

$$F = ma \tag{12}$$

or some of the kinematics equations you have seen before such as

$$v_f = v_i + at \tag{13}$$

$$v_f^2 = v_i^2 + 2a\Delta x \tag{14}$$

$$\Delta x = v_i t + \frac{1}{2}at^2 \tag{15}$$

$$\Delta x = \frac{1}{2}(v_i + v_f)t. \tag{16}$$

Such equations enable us to predict what will happen give some set of *initial conditions*. If we know what the initial velocity, the acceleration, and the time an object travels for, then we can figure out its final velocity.

Now the above equations can actually be more generally formulated as something known as a **differential equation**. I won't go too in-depth on the mathematical properties of a differential equation, but instead introduce it somewhat schematically / pictorially. A differential equation essentially describes how the rate of change of a certain variable is altered in relation to other physical quantities of interest. This wishy-washy definition may feel very unclear and abstract, so let's look at a few examples.

Let's look at the classic equation $F = ma$, force equals mass times acceleration. We can instead reformulate it as a differential equation of the following form

$$F = m\frac{dv}{dt} \tag{17}$$

where we've replaced the acceleration $a$ with the term $\frac{dv}{dt}$, the time derivative of velocity. The $\frac{d}{dt}$ term is a time derivative, and when we apply it to the velocity, we are essentially asking how much does the velocity change for a small change in time. This may feel very familiar, as you have heard acceleration described to be a change in velocity over time, or

$$a = \frac{v_f - v_i}{t}. \tag{18}$$

The derivative expression is just a more general way of describing this. Let's think about how the term $\frac{dv}{dt}$ works before moving on. If my velocity changes a lot within the allotted time-frame, then $\frac{dv}{dt}$ will be large. If it changes very slowly, then it will be small. This intuitively follows along with our understanding of acceleration: if something very quickly gains a lot of speed within a short time, it is accelerating very quickly.

Note that we can look at lots of different derivatives, not just derivatives in time. For example, $\frac{dm}{dx}$ the change in mass for some small change in position, could be useful in analyzing the mass of an object that isn't uniform. $\frac{dV}{dT}$, the change in volume with a change in temperature, could be useful in describing how an objects volume deforms or morphs when we turn the temperature up or down.

Let's now analyze everything in the context of our differential equation for Newton's law, as described in equation 17. Remember, earlier I said that a differential equation describes how the rate of change of something is altered in relation to other physical variables. In the case of the equation

$$F = m\frac{dv}{dt} \tag{19}$$

the rate of change we are looking at is the velocity, and the other physical variables we are concerned with are $F$ and $m$, the force and the mass. To say this more naturally, Newton's second law describes how the rate of change of a body (let's say, for example, that the body is a ball) changes in response to the force applied upon it or its mass.

Let's investigate how the rate of change of the velocity differs when we alter our physical variables. If we fix the mass of the ball but increase $F$, we see that the rate of change of velocity $\frac{dv}{dt}$ should increase. This makes sense. If I push the ball harder, it quickly changes to a much faster speed.

On the other hand, if I fix how hard I push the ball but make it heavier, then $\frac{dv}{dt}$ will decrease. This also makes sense: if I push a heavier ball and a lighter ball with the same amount of force, the heavier ball will end up at a lower top speed than the lighter ball.

All of that information is encapsulated within the differential equation, and we see that differential equations give us a way to analyze how the rate of change of one variables depends on other variables.

By solving the differential equation (which we will later do for a quantum case), we can actually figure out how the variable we are interested in analyzing (in this case the velocity) evolves over time! Differential equations are a powerful modeling formalism for predicting how systems will change given some set of initial conditions. The kinematics equations I described above actually all arise from differential equations.

I'll provide a couple more illustrative examples below. Try and figure out what they mean.

### 4.3.2 Some Example Differential Equations

**Heat transport**

When a hot object comes in contact with a cold object and heat flows from the hot object to the cold object, what is the rate of change of the temperature of the cold object as a function of position?

$$q = -k\frac{dT(x)}{dx} \tag{20}$$

where $q$ is rate of heat flow and $k$ is the thermal conductivity of the material (how much heat is transferred by the object when the temperature changes) and $\frac{dT}{dx}$ describes the change in temperature.

**Radioactive decay**

When we have a sample of particles undergoing radioactive decay, how many particles are left after a certain amount of time?

$$\frac{dN(t)}{dt} = -\lambda N(t) \tag{21}$$

where $\lambda$ describes the activity of the sample, $N$ describes the number of particles in the sample, and $\frac{dN}{dt}$ describes how the amount of particles in the sample decrease or increase over time.

**Quantum Mechanics!!!**

If I have some quantum particle near something else that attracts it (pulls it inward), how does the probability of finding the particle vary at different locations?

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{22}$$

where $\hbar$ is Planck's constant, $V(x)$ is that "something" that is attracting the particle, $E$ is the energy in the system, and $\psi(x)$ is something mysterious we will discuss in a second. This is the famous *Schrodinger's Equation*. For the time being, I will just plop it here without much exposition, but we will now begin to slowly break it down.

## 4.4 The Wavefunction

So what is this mysterious $\psi$ symbol we saw in the Schrodingers equation? This is a wave-function, a special object that describes all the characteristics of our quantum state. For example, an electron held in a box might have a wave-function that tells us where it might be in the box. An atom moving around in a gas might have a wave-function that tells us where it might be at a certain time.

When early theorists of quantum mechanics were coming up with the mathematical machinery to describe quantum mechanics, they were aware of some of the odd experimental results and fundamental properties of quantum mechanics and quantum systems. Thus, in order to postulate the mathematics that properly modeled quantum systems, they needed to be able to encapsulate all of these strange phenomena, such as the fact that quantum mechanics is probabilistic and has some inherent uncertainty to it. The wave-function is the tool they made up to describe such a thing, and there are actually a lot of debates on whether the wave-function is something that is actually physically observable or just a neat math trick we use to do quantum mechanics. Here, we won't go too deep into the mathematics, and I will attempt to give you a good understanding through examples and plots.

To make things a bit more clear, let's consider a specific example. Consider a quantum system where I have an electron confined in a box as shown in figure 13. Let's say I

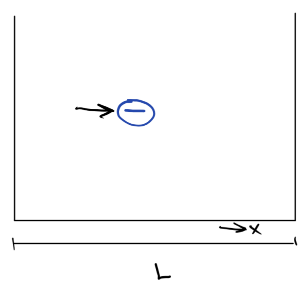


Figure 13: An electron (blue ball) confined in an imaginary box of length $L$. The electron is only allowed to move in one dimension (the x dimension) and the box can be thought of as two hard walls. The electron can bounce off the walls.

am doing an experiment. I place the electron in the middle of the box and give it a push to the right, then I start a timer. After five seconds, I measure the position of the electron. Due to Heisenberg uncertainty, as discussed in section blank, we know that we have to make this measurement many many times and make an average to get an actual useful answer.

Let's now think about what the wave-function for the electron needs to describe. The primary characteristic I want to know is the position of the electron, so the wave function must be a function of $x$, say $\psi(x)$. The wave-function must also encapsulate the probabilistic nature of quantum mechanics. I do my experiment 100 times: 80% of the time, my experiment results tell me that the electron will be at position 1, while 10% of the time I measure it at position 2, then 5% of the time I get a measurement at position 5... Hmmm... does this sound like something that's kind of familiar?

You might be thinking of a probability distribution. Its almost like someone who is playing darts: 5% of the time, the dart hits the bullseye, 10% of the time, they hit the inner ring, 10% of the time, they hit the next ring out, and so on and so forth. To describe such results, we draw things like probability distributions. You may have seen them used to describe the number of people that get a certain score on a test, or how tall people are, or how much people weigh. Below are a few examples:

So the wave-function for this example should be something similar to a probability distribution that describes where you might expect to find the electron when we take a measurement. Naively, we might expect the wavefunction to look like that of figure 14.

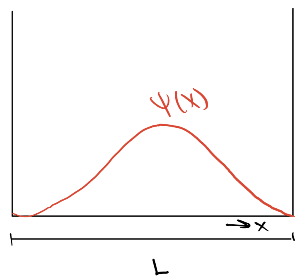


Figure 14: A naive guess at the wave-function of the electron in a box.

Now why is that? Well, when we give the electron a push, it travels to the right, meanders around for a bit, then bounces off the right wall and comes back to the middle. It then probably meanders for a bit, then bounces off the left wall, and returns to the middle. We see that for every bounce-back event at a wall, the electron has to travel through the middle region of the box twice, so it is more likely to be found in the middle of the box than in the sides. Indeed, if we do out the math, applying wave-functions and the mysterious Schrodinger's equation, we do see that result!

#### 4.4.1 Properties of the Wave-function

So now that we have a bit of a schematic overview of what a wave-function is, let's be a bit more explicit and talk about some of its properties. Though I drew a parallel to probability distributions, you should definitely be aware that a wave-function is quite different from a classical probability distribution. It is just a nice simplified picture to think about for clarity. Examining the wave-function at a specific location and a specific time does not exactly give us the probability of finding the electron in that location at that time. To figure out such things out, we need some slightly more complex mathematics.

First off, the probability density $\rho(x,t)$ is equal to the norm-squared of the wave-function. Mathematically, this looks like

$$|\psi(x,t)|^2 = \rho(x). \tag{23}$$

The probability density $\rho(x)$ basically gives us a plot similar to those basic distributions I talked about above, and so we now have extracted some physical information from the wave-function! From the probability density, we can then extract the probability of finding the particle at some specific region.

If those terms are a bit unfamiliar to you, don't worry, things will become more clear later. The first question you have is, what is the norm-squared (the pipe looking symbols squared, $||^2$ symbol for? Well, the wave-function consists of imaginary parts a lot of times, and by applying that $||^2$ operation, we essentially only keep the real part, which is the part we can physically observe. You may be wondering: why didn't the developers of quantum mechanics just make the wave-function itself real? Why jump through all these hoops? If you continue with quantum mechanics you will see all sorts of reasons for why that is the case, but I will not go off on that tangent.

Now that we have a probability density, we can then find out the probability of the particle being in a specific location over the course of the experiment. To do so, we just have to look at a slice of area within the probability density. Pictorially, for some random wave-function, the probability of finding the particle between points $a$ and $b$ are shown in the shaded region in figure 15.

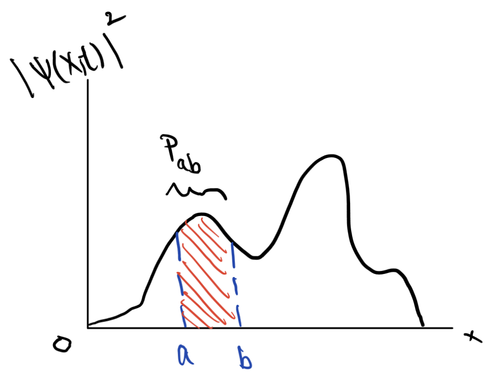


Figure 15: Plot of $|\psi(x)|^2$. The shaded region denotes the probability $P_{ab}$ of finding the particle between points $a$ and $b$.

In math terms, what this means is we have to take an integral. Thus, the rather long-winded way in which we can extract probabilities from the wave-function is by first taking the norm-squared and then integrating over your region of interest. So let's say you have a box stretching from $x = 0$ to $x = 5$, and you want to know the probability of finding the particle between $x = 1$ and $x = 2$, then we would compute

$$\text{Probability of electron being between } x = 1 \text{ and } x = 2 = \int_1^2 |\psi(x,t)|^2 dx. \tag{24}$$

Having jumped through so many hoops, you may now want to know what the wave-function itself actually is. Sure its norm-squared gives a density, and then integrating that gives us a probability, but what is the wave-function itself? Unfortunately, I am going to cop-out and just not tell you. Perhaps it can serve as motivation for you to take more physics.

An additional property of the wave-function we'll want to keep in mind is something known as continuity. If we imagine the wavefunction as a line, then it can't sporadically be broken. An example of this is shown in figure 16.

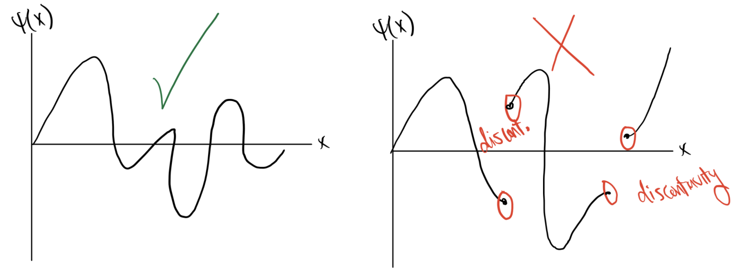


Figure 16: An example of a continuous (green check) and discontinuous (red cross) wavefunction

Regardless of what's going on in your quantum system, the wave-function can just sporadically jump to 0 or shift to another value. It can really quickly decay or increase to that value, but it can't just hop around. More examples are shown in figure 17. Why

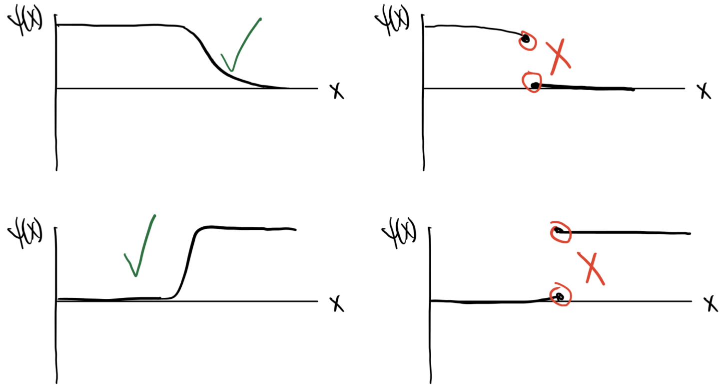


Figure 17: Some more examples of continuous and discontinuous wave-functions.

is this the case? Well one easy argument is to think in terms of the probability density $|\psi(x)|^2$. We mentioned early that $|\psi(x)|^2$ is a quantity that we can actually physically observe and measure. We also said looking at the area under $|\psi(x)|^2$between points $a$ and $b$ gives us the probability $P_{ab}$ of finding the electron there. However, if there is a discontinuity of the wave-function and it suddenly jumps somewhere between points $a$ and $b$, then what does $|\psi(x)|^2$ even mean? What does the area under that curve even mean? It all becomes nonsensical and our theory falls apart. And so you can think of continuity as something the theory requires, and indeed this fact has been confirmed by experiments.

### 4.4.2 Some more mathematics

I'll show you some quick properties of the wave-function that really relate them to classical probability and statistics theory that you have seen before. If you don't too much about math like this, just skip this stuff.

The norm-squared of the wavefunction is just a probability density, so it obeys all the same rules as a usual probability density from statistics:

$$\textbf{Normalization:} \int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1. \tag{25}$$

$$\textbf{Expectation values:} \langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x,t)|^2 dx. \tag{26}$$

$$\textbf{Variance:} \sigma^2 = \langle \Delta x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2. \tag{27}$$

All of the same properties you know and love apply in the same way!

Quantum mechanics is a very statistical theory and indeed all of our favorite classical statistical properties apply in the same way!

### 4.4.3 Projective Measurement

**(**Experiment: Tracking the Collapse of the Wave-function**)**

An extremely important feature of quantum mechanics that I glossed over are the effects projective measurement on a quantum state. Performing a type of measurement known as a projective measurement on a wavefunction (that initially looks fairly broad and featured) collapses it to a sharp and pointy distribution known as an eigenstate. This process is shown in figure 18.

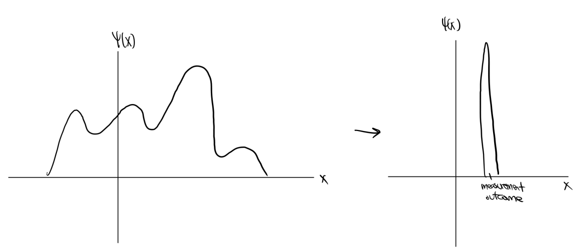


Figure 18: The collapse of the wavefunction to an eigenstate following a projective measurement.

Let's think about this one more conceptually. Let's say we have a wavefunction that describes the positions of a particle. What this wavefunction allows me to physically infer is that there are various probabilities of finding the particle in different positions.

Some are more likely, some are less likely: the wave-function tells me the general spread of things and gives me information about where the particle could be.

However, once I perform a projective measurement, I have FOUND OUT where the particle actually is. Now that I know where the particle is, then the state of the particle is no longer probabilistic. I now KNOW it is in the position I have measured. The wavefunction is subsequently collapsed to a sharp peak centered at that point.

## 4.5 The Schrodinger Equation

A big part of physics is making predictions on how systems will evolve when we change certain quantities. I mean, that's the whole purpose of experiments after all! We ask questions like how long it takes a object particle to fall to the ground compared to a smaller object, or how much electricity will run through your body if you stick a fork in an outlet, or how far a ball will roll on a ramp with a certain amount of friction. Quantum mechanics is no different, except the chief quantity we want to make predictions about is the wave-function $\psi(x)$ that we just talked about.

Earlier, in section 4.3.2 I introduced something known as the Schrodinger Equation, a differential equation that describes how the wave-function $\psi(x)$ changes with respect to a couple of different physical quantities. The Schrodinger equation basically gives us a framework to analyze evolution of the wave-function in a similar way to all of those classical physics examples I just talked about!

Now that you're an expert on differential equations and the wave-function $\psi(x)$, you can perhaps interpret what this thing is saying.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{28}$$

Ok, I was lying. This equation probably still looks a bit monstrous and confusing, so let's break it down a bit more. Let me first list to you what the variables and symbols mean: $\hbar$ is once again the reduced Planck's constant, we are doing quantum mechanics after all, $m$ is the mass of the particle we are analyzing, $\psi(x)$ denotes the wave-function of the particle we are investigating, $V(x)$ denotes some potential, $E$ denotes the energy of the system, and $\frac{d^2}{dx^2}$ denotes the second derivative (this one is probably a bit confusing to you, and rest assured we will go over it in more detail later).

Why do we have $V(x)$ and $E$ here? Well, these are just quantities that we are usually interested in when we're doing experiments or modeling quantum systems. $V(x)$, the potential, is usually something that can interact with the particle. It will perhaps pull in the particle when it gets really close, or push it away. Its basically something that can lead to interesting dynamics, like a little magnet or a source of electricity. $E$ just describes the energy in the system.

The second derivative might be a bit unfamiliar, but it is just two first derivatives taken successively, or

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx}. \tag{29}$$

You can think of this as the rate of change of the rate of change. You may have learnt that the time derivative of the position, or basically how much something has moved within a specific increment of time is the velocity, or $\frac{dx}{dt} = v$ (think of a car driving one mile in one minute). You might also have learnt that the time derivative of the velocity, or how much the velocity has changed in a specific increment of time is the acceleration, or $\frac{dv}{dt} = a$ (think of a car accelerating from a speed of 30 miles / hour to 60 miles / hour in five seconds). We then see that if we use the second derivative, we can relate the acceleration to the position:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{dv}{dt} = a. \tag{30}$$

In plain English, we can say that the rate of change of the rate of change of position gives the acceleration. In the case of the Schrodinger equation, we are taking a derivative with respect to the position $x$, and so the term $\frac{d^2\psi(x)}{dx^2}$ just means the rate of change of the rate of change of the wave-function with some change in position. If this is all a bit confusing (which it likely is) you can just think of it as something that is related to how much and how quickly the wave-function changes when we move from one position to another.

In a similar way to Newton's law, which enables us to predict how a physical system like balls and springs will evolve over time, Schrodinger's equation allows us to predict how the wave-function evolves over time. Whereas in the classical case, the parameters of interest were mass and force, here, we are concerned with the potential and the total energy of the system.

You still may not really understand what the Schrodinger equation is describing, or why it even is useful, and that's perfectly alright. Its a very complex topic that requires a good bit of mathematical machinery, but if you take anything away, its that it is basically the framework for describing how the wave-function evolves in a quantum system. I can design where I put a magnet in a box with an electron, and then proceed to solve for how the wave-function will change along the length of the box. From that, I can then compute the probability of finding the electron in certain regions, as well as many other useful physical quantities.

In fact, I can set up all sorts of experiments and try to solve for them. I could make my box donut shaped and put magnets every two micrometers apart. I could shoot protons into the box, or make the magnet a really weird shape. That is why the Schrodinger equation is so important: it allows us to look at the behavior of quantum systems, and correctly models all the strange attributes of quantum mechanics we have talked about so far. We'll look at an example of applying the Schrodinger equation in a second.

# 5 Quantum Tunneling

In this section, we'll examine a fun phenomena you may have heard about known as quantum tunneling. You may have heard about this phenomena in things like popular science magazines or even in movies. Let's first imagine a classical situation: Let's say I throw a tennis ball at a cement wall. Now, I ask you "What is the probability the tennis ball passes through the wall?" and if you are not completely insane, you will say that there is zero probability that the tennis ball passes through. I will then tell you that you are indeed correct.

Now let's imagine a different situation. I throw an electron at a very very strong potential, which you can effectively think of as concrete wall. I turn to you again and ask "What is the probability the electron passes through the wall?" and being the sane person (as we previously found out) you are, you say that there is zero probability that the electron is found outside. However, despite the fact that you are sane, physics is insane, and physics actually decrees that there is a NONZERO PROBABILITY the electron passes through the wall. This is the phenomena known as **quantum tunneling**. A quantum particle actually has a nonzero probability of tunneling through a barrier or wall it should not be able to get through (though this probability is usually very very small). In the next couple of subsections, we will both mathematically derive and schematically explain why tunneling arises, and some interesting consequences and applications of the effect.

## 5.1 Understanding Tunneling

As promised we will consider this problem through the context of the Schrodinger equation. Throughout this process, we will apply two of the central tenants of physics: simplification and approximation.

Recall the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{31}$$

Note that I have written out the Schrodinger equation that is only dependent on the $x$ direction. You might be thinking that we also need the $y$ and $z$ direction since the electron won't fly perfectly straight and may bob up and down or back and forth, but for the sake of simplicity, we will pretend that the electron travels perfectly horizontally. This will be sufficient for us to understand the physics of the situation. Remember, it is always best to solve the simplest problem first!

Here, the energy $E$ is just the energy of the electron. Our goal is to solve the differential equation (the Schrodinger Equation) and get an expression for $\psi(x)$ as a function of the energy $E$ and the potential $V$. Before proceeding, one of the critical components we must model is the potential $V(x)$. For our quantum system, we have an electron flying towards a wall, which we will say has some thickness $L$. The electron can't be the potential, so the box has to be, but how does a wall act like a potential?

You may have heard about things like gravitational potentials, spring potentials, or electric potentials. Those examples are pretty much what $V(x)$ is. In fact, in classical mechanics, you might have treated a problem that asks you to find how much energy a ball needs to climb up a hill, as illustrated in figure 19.

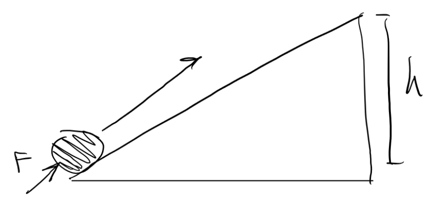


Figure 19: Classical mechanics problem of finding how big of a push is required to climb up a hill.

In such a problem, you have to figure out how fast the ball needs to go (aka how much kinetic energy $KE = \frac{1}{2}mv^2$, where $m$ is the mass of the ball and $v$ is the velocity of the ball) to climb up a ramp with some height $h$ (which has some potential energy barrier $PE = mgh$ where $g$ is gravity). If $KE > PE$, meaning the ball has enough kinetic energy to climb up the potential energy hill, then it can successfully get to the top.

We can think about the wall in a very similar way, pretty much like a really big hill the ball has to climb. However, physically, it seems like no matter how fast the ball is going, it won't be able to climb this hill. Its so steep and tall, there's no way the ball can get up there! Thus, to model this scenario, we will just say that where the wall is, there is just some energy $V_0$ that is greater than the energy $E$ of the particle. However, in the region between the walls, the ball is free to fly as it pleases.

Now that we have figured out how to model the potential, we can now draw out our experimental set-up, which is shown in figure 20. With this in mind, we can write the

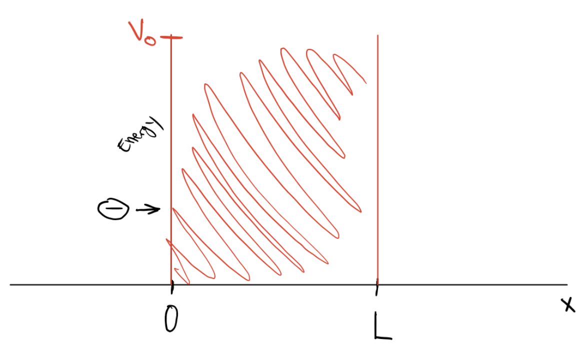


Figure 20: A pictorial model of an electron flying towards a potential barrier.

potential V(x) in the following way:

$$V = 0 \quad \text{at } x < 0 \tag{32}$$
$$V = V_0 \quad \text{between } x \geq 0 \text{ and } x \leq L \tag{33}$$
$$V = 0 \quad \text{at } x > L. \tag{34}$$

Perfect. Let's now divide our experiment into regions 1 (left of the barrier), 2 (inside the barrier), and 3 (right of the barrier), as shown in figure 21. This will help us organize things in a bit.

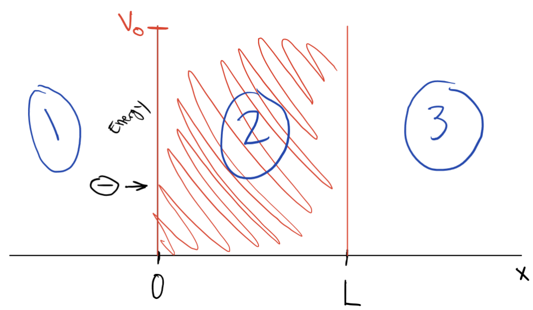


Figure 21: Electron in a box divided into three regions.

Mathematically, the region to the left of the barrier (region 1) will be described by $x \in (-\infty, 0)$ (these symbols mean that $x$ can be found in the region between $-\infty$ and 0), the region in the barrier (region 2) will be $x \in [0, L]$, and the region outside of the barrier (region 3) will be $x \in (L, \infty)$.

Ok, unfortunately we have to start doing some math now. However, some things might be a little to advanced for you to understand at the moment, so if you don't know why the math works out the way it does, you'll just have to trust me. Don't worry though, along the way I will attempt to give as much exposition as possible to make things clear.

We split our box into three regions, and similarly, we will split our wave-function into three regions. Now, we have some math to do.

Let's start with region 1, before the electron has arrived at the barrier. We know that $V = 0$, so the Schrodinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x). \tag{35}$$

We can shift some stuff around to make things easier. Also, going forwards, I will also just write $\psi(x)$ as $\psi$ since I'm lazy.

$$\frac{d^2\psi_1}{dx^2} = -\frac{2mE}{\hbar^2}\psi_1 \tag{36}$$

We will have the same equation for region 3, since the potential is also $V = 0$:

$$\frac{d^2\psi_3}{dx^2} = -\frac{2mE}{\hbar^2}\psi_3. \tag{37}$$

However, in region 2, the potential is nonzero, so we have to include that term in the Schrodinger equation. $V = V_0$, so then things look like

$$\frac{d^2\psi_2}{dx^2} = (-\frac{2mE}{\hbar^2} - V_0)\psi_2. \tag{38}$$

In summary, we have the following Schrodinger equations for each of our equations:

$$\begin{cases} \frac{d^2\psi_1}{dx^2} = -\frac{2mE}{\hbar^2}\psi_1 & \text{when } x < 0 \\ \frac{d^2\psi_2}{dx^2} = (-\frac{2mE}{\hbar^2} - V_0)\psi_2 & \text{when } x \geq 0 \text{ and } x \leq L \\ \frac{d^2\psi_3}{dx^2} = -\frac{2mE}{\hbar^2}\psi_3 & \text{when } x > L \end{cases}. \tag{39}$$

Yikes, things are looking hairy. We have a system of differential equations. How could we over-complicate things in such a way? Don't worry, we'll deal with them one at a time. Let's focus on the first region:

$$\frac{d^2\psi_1}{dx^2} = -\frac{2mE}{\hbar^2}\psi_1. \tag{40}$$

For simplicity, I will designate a new variable $k = \frac{\sqrt{2mE}}{\hbar}$. I can do this since $m$, $E$, and $\hbar$ are all just constants in this expression. None of them really vary with $x$, so I can just pack them all into a more simple constant. $\psi_1$ is the only expression we can't touch, since it does vary with $x$ and we also need to take its derivative. Our equation then becomes

$$\frac{d^2\psi_1}{dx^2} = -k^2\psi_1. \tag{41}$$

We can solve this equation with our new variable $k$ and then at the end, just substitute in its value. This is a common technique we use to simplify things.

We are trying to find $\psi_1$ that satisfies equation 40. We shall apply a powerful tool for doing math and physics questions: guessing!

Remember the $\frac{d^2}{dx^2}$ symbol just means to take the derivative of $\psi_1$ with respect to $x$ twice. $\psi_1$ will be some function of $x$. Maybe its $\psi_1 = x^3 + x^2$, or $\psi_1 = sin^2 x + 3$. However, from the structure of the equation, I will make the guess that $\psi_1$ is described by an exponential expression

$$\psi_1 = Ae^{ikx} + Be^{-ikx} \tag{42}$$

where $A$ and $B$ are just some constant values we will figure out later and $i$ is the imaginary number.

You may be wondering why the heck I made such a guess. The reason is actually quite intuitive. Let's just look over the derivative of a simpler expression $a(x) = e^{ikx}$.

Remember that when you take a derivative of an exponential, you just multiply the same exponential by all of the constants in front of the variable you are taking the derivative of, which in our case is $x$. We see that

$$\frac{da(x)}{dx} = \frac{de^{ikx}}{dx} = ike^{ikx}. \tag{43}$$

Now, the Schrodinger equation involves a second derivative, so let's do it again:

$$\frac{d}{dx}\frac{da(x)}{dx} = \frac{d(ike^{ikx})}{dx} = (ik)^2 e^{ikx} = -k^2 e^{ikx} = -k^2 a(x). \tag{44}$$

Whoa, wait a second. Doesn't that kind of look like the Schrodinger equation if we just replace $a(x)$ with $\psi(x)$? That's why we use exponentials in our guess. Now you still might be wondering, why do we needs the coefficients $A$ and $B$, and why do we have an exponential that is $e^{ikx}$ and another one that is $e^{-ikx}$? Why are we overcomplicating things? These additional terms we can reason through by considering what is physically occurring when the electron arrives at the wall.

Let's think about what events will occur in region 1. Of course, we will have the electron propagating from left to right towards the potential wall. This event we will give the label of $Ae^{ikx}$. We also need to remember that the electron will bounce off of the potential wall and actually travel from right to left back towards where it came from. This event we will give the label of $Ae^{-ikx}$. All of the events are laid out more clearly in figure 22.

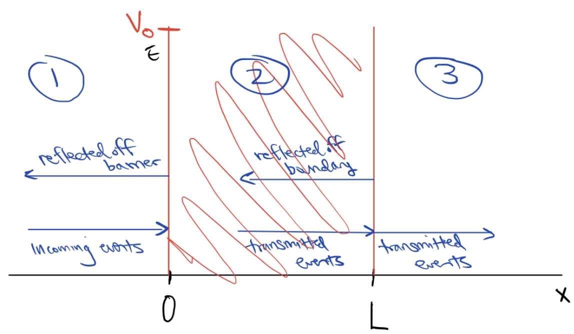


Figure 22: Schematic representation of the events that can occur at each boundary. At the first boundary where the electron meets the barrier, it may pass through or it may just bounce off. At the second interface between the barrier and free space, the electron could once again just bounce off or pass through.

The different signs of the terms in the exponent represent the different directions the electron is traveling in. The coefficients $A$ and $B$ also basically act like weights for these events. $A$ is an analog for the amount of events where the electron traveling from left to right, while $B$ is an analog for the amount of events where the electron travels from right to left after bouncing off the wall. This is why we make the guess that

$\psi_1 = Ae^{ikx} + Be^{-ikx}$. We will have a similar result for $\psi_3$. However, there is nothing to reflect off of in region 3, so we will only have one term, the left to right moving term. Furthermore, the constant will also be different, as the amount of events will be different. For region 3, we thus have

$$\psi_3 = Fe^{ikx}. \tag{45}$$

Ok, so we've made some guesses for the wave-function in regions 1 and 3. What about the more complex region 2? We will do the same thing and designate a new variable $j = -\frac{2m}{\hbar^2}(E - V_0)$. This enables us to rewrite our equation to be

$$\frac{d^2\psi_2}{dx^2} = (-\frac{2mE}{\hbar^2} - V_0)\psi_2 \tag{46}$$

$$\downarrow \tag{47}$$

$$\frac{d^2\psi_2}{dx^2} = j^2\psi_2.n \tag{48}$$

Applying our guessing technique again, we guess

$$\psi_2 = Ce^{-jx} + De^{jx} \tag{49}$$

where $C$ and $D$ are constants we will figure out later. Note that we have two terms here once again, as at $x = L$, we have another "wall" where we transition from the infinite barrier to regular free space. Another sneaky thing I have done is that I do not include the imaginary $i$ here, as we have positive $j^2$ on the right-hand side of 48 instead of $-k^2$ like we had in equation 41. We included $i$ previously since $i^2 = -1$, which gives us the negative we needed for $-k^2$, but since $j^2$ is positive, we don't need $i$ in our guess. I'm not going to explain why we do things this way because its not that relevant. Sorry.

Now that we have a description for all of our wave-functions in each region, we can go back to our diagram of the different events and label them with our derived expressions that represent them (shown in figure 23).

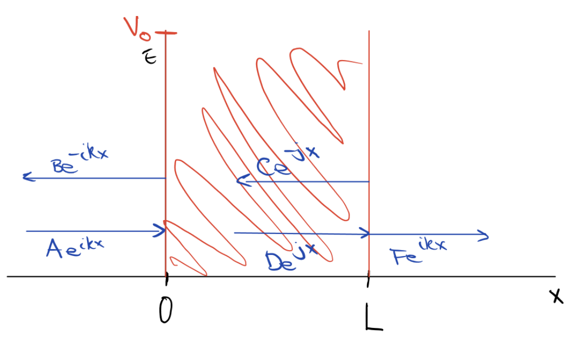


Figure 23: Each event that occurs at the boundaries now labeled with their corresponding mathematical expressions.

All of the terms with the positive exponential, such as $Ae^{ikx}$, $De^{ikx}$, and $Fe^{ikx}$ represent events where the electron continues to move from left to right (which we will call transmission events) and all of the terms with negative exponentials, such as $Be^{-ikx}$ and $Ce^{-ijx}$ represent events where the electron bounces off of a boundary (which we will call reflection events).

Ok, we've made some guesses for $\psi$ in each region, and we have that

$$\begin{cases} \psi_1 = Ae^{ikx} + Be^{-ikx} & \text{when } x < 0 \\ \psi_2 = Ce^{-jx} + De^{jx} & \text{when } x \geq 0 \text{ and } x \leq L \\ \psi_3 = Fe^{ikx} & \text{when } x > L \end{cases} . \tag{50}$$

We need to solve for $A$, $B$, $C$, $D$, and $F$. Now you may be thinking I'm an idiot and this is unapproachable. I mean, high school algebra probably has drilled into you that for five unknowns, you need to have five equations, and from what your eyes are telling you here, we have no equations that relate these constants, just a bunch of expressions. However, there is one condition we can apply that will introduce a lot more equations for us: continuity of the wave-function. I alluded to this in 4.4.1. Our wave-function can't just jump to another value once they arrive at a different region, they must smoothly change. This means that at the point where regions transition to other regions ($x = 0$ and $x = L$), the wave-functions in both regions must be equal. Mathematically, this means that

$$\psi_1(0) = \psi_2(0) \tag{51}$$

$$\psi_2(L) = \psi_3(L). \tag{52}$$

Furthermore, if the functions themselves are equal at those points, the derivatives must also be equal. This gets us two more conditions

$$\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \tag{53}$$

$$\frac{d\psi_2(L)}{dx} = \frac{d\psi_3(L)}{dx}. \tag{54}$$

Now we have four equations and five unknowns. Let's list all of them out

$$\psi_1(0) = \psi_2(0) \rightarrow \qquad A + B = C + D \tag{55}$$

$$\psi_2(L) = \psi_3(L) \rightarrow \qquad Ce^{-jL} + De^{jL} = Fe^{jL} \tag{56}$$

$$\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \rightarrow \qquad ikA - ikB = -jC + jD \tag{57}$$

$$\frac{d\psi_2(L)}{dx} = \frac{d\psi_3(L)}{dx} \rightarrow \qquad j(-Ce^{-ijL} + De^{ijL}) = ikFe^{ikL}. \tag{58}$$

So we're pretty close but not quite there. We need one more equation or one less variable. Let's dig our head out of this math hole and think about the physics again.

The quantum mechanical process we're trying to investigate is tunneling. We want to know how probable it is for the electron to make it into region three. Schematically, this will be given by the following expression:

$$\text{Tunneling Probability} = \frac{\text{Events where electron is found in region three}}{\text{Total number of experiments}} \tag{59}$$

Remember that the probability of events is tied to the norm-squared of the wavefunction, or $|\psi|^2$. Furthermore, our experiment begins in region one, so the total number will just be the number of events corresponding to the electron traveling from left to right in region 1. This expression is given by:

$$\text{Total number of experiments} = |Ae^{ikx}|^2 = |A|^2. \tag{60}$$

I will omit the details on why the expression evaluates to be this. You'll just have to trust me.

You can probably now infer that the events where the electron is found in region three is given to be

$$\text{Events where electron is found in region three} = |Fe^{ikx}|^2 = |F|^2. \tag{61}$$

So the tunneling probability is just

$$P_{tunneling} = \frac{|F|^2}{|A|^2}. \tag{62}$$

Ok cool, we figured out the actual expression we need to find to get a solution. How does this simplify our system of equations? Well, since we are looking for $\frac{F}{A}$, we can divide all of our expression by $A$ and instead solve for $\frac{F}{A}$. This enables us to eliminate an additional variable. Our system of equations is now:

$$1 + \frac{B}{A} = \frac{C}{A} + \frac{D}{A} \tag{63}$$

$$\frac{C}{A}e^{-jL} + \frac{D}{A}e^{jl} = \frac{F}{A}e^{jl} \tag{64}$$

$$ik - ik\frac{B}{A} = -j\frac{C}{A} + j\frac{D}{A} \tag{65}$$

$$j(-\frac{C}{A}e^{-ijL} + De^{ijL}) = ik\frac{F}{A}e^{ikL}. \tag{66}$$

You may be a bit confused and wondering to yourself "the $A$ is still there! What have we accomplished?" but the way you should think about it is that we have essentially redefined the variables of interest in our system of equations. Instead of solving for five variables $A$, $B$, $C$, $D$, $F$, we only need to solve for four: $\frac{B}{A}$, $\frac{C}{A}$, $\frac{D}{A}$, and $\frac{F}{A}$.

So here's the awkward part. We've successfully simplified our expressions to be solvable. However, actually doing the algebra isn't really that easy, and I can't really give you a simple satisfying result. Since I value my time, I'm just going to feed this system of equations to a computer and ask for an answer for $\frac{F}{A}$ and then $\frac{|F|^2}{|A|^2}$.

The computer does the heavy lifting, yielding

$$\frac{F}{A} = \frac{e^{-ikL}}{\cosh jL + i(j/2k - k/2j)\sinh jL} \tag{67}$$

$$P_{tunneling} = \frac{|F|^2}{|A|^2} = \frac{1}{\cosh^2 jL + (j/2k - k/2j)^2 \sinh^2(jL)} \tag{68}$$

where cosh and sinh are these things called hyperbolic cosine and hyperbolic sine, belonging to a class of expressions known as hyperbolic trigonometric functions (which is not something you need to know right now). cosh and sinh are given by

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{69}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{70}$$

Yikes. That's a horrific expression. However, that's not even the full form. Remember earlier we made some simplifying substitutions $k = \frac{\sqrt{2mE}}{\hbar}$, $j = -\frac{2m}{\hbar^2}(E - V_0)$. I'm not going to substitute those back in, but you get the point. Its pretty ugly.

Luckily, with some math tricks (which I won't share with you), we can simplify it down to

$$P_{tunneling} = 16\frac{E}{V}(1 - \frac{E}{V})e^{-2jL}. \tag{71}$$

Phew, that was a long journey to finally get a solution. If you didn't really follow things that well, don't worry about it too much, its not a huge deal. What's important here is the expression for $P_{tunneling}$ we have found. Notice that it is indeed nonzero! So the electron actually has some small probability of making it through the wall!

This effect is a result of the continuity of the wave-function that we discussed earlier. Since the wave-function can't instantly jump to zero once a wall shows up, it quickly decays, yielding a small but nonzero probability of passing through the wall.

Let's make some plots to examine how the tunneling probability changes when we alter different parameters. In the expression, $E$ will typically be a fixed value, and we will vary the strength of the potential $V$ and the length of the wall $L$. Since we aren't really concerned about getting any exact numbers, I will just set all of the variables and constants we don't really care about (such as $E$, $\hbar$, and $m$) to 1 and vary $V$ and $L$. This will provide us with some information about how rapidly the tunneling probability decreases as we increase the strength of the potential and the length of the wall. Doing so yields the graphs shown in figure 24. We see that increasing the strength of the potential (making the wall taller) decreases the tunneling probability as expected. We also see that increasing the thickness of the wall $L$ also decreases the tunneling probability, which is also something we would expect. The tunneling probability also seems to be more sensitive to changes in the length, as it crashes down to around $10^{-280}$ at a length of around $2 \times 10^1 = 20$, whereas it reaches a similar value at around $V = 190$. Overall, we see that the taller and thicker the wall, the smaller the probability of the electron tunneling through, with the behavior being especially sensitive to thickness.

Another variable we can consider is the energy. What do we expect here? If the electron is more energetic, then it should have a higher tunneling probability. If I throw a ball harder at a wall, I should expect a higher probability of it going through the wall. Wait, that's not quite right is it... Oh well, you get the point.

Plotting out the tunneling probability as a function of the energy yields the results in

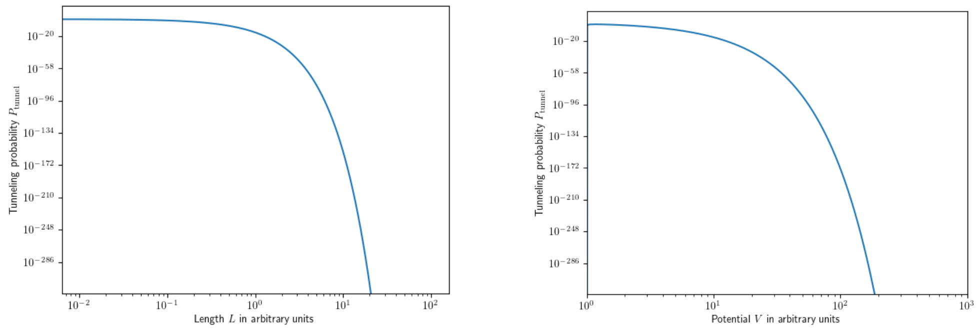


Figure 24: Plot of tunneling probability $P_{tunnel}$ vs. length $L$ (left) and $P_{tunnel}$ vs. potential $V$ (right). Both plots are in logarithmic scale. In the left plot, constants such as $\hbar$ and $m$ have been set to 1, and $V$ has been set to 10. In the right plot, $\hbar$, $m$, and $E$ are all set to 1. The units for these plots are thus arbitrary and should not be interpreted numerically. These plots just serve to provide some intuition on behavior.

figure 25. So we see that as energy increases, tunneling probability increases like we

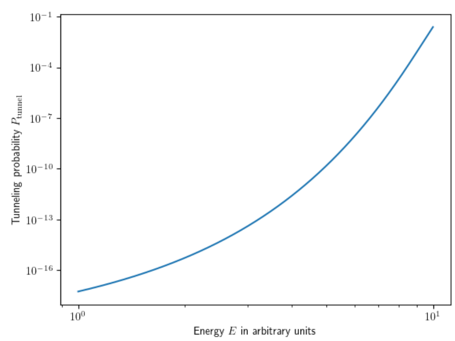


Figure 25: Logarithmic plot of energy of particle vs tunneling probability. $\hbar$, $L$, and $m$ have been set to 1, $V$ has been set to 10.

would expect. Now one final detail that I have omitted in these plots (to avoid overcomplicating things) is the question of what happens when the energy of the particle $E$ and the height of the wall/barrier potential $V$ are equal. Well, what do you expect? If the particle is more energetic the it just means that it has enough energy to make it through the barrier, and so it will always pass through. Pretty interesting stuff!

### 5.1.1 Example: Battery Leakage

However, its hard to really get a feel for things without some actual real-world numbers. We will analyze electrons, which conveniently have a unit of energy associated with them, the electron volt. Experiments involving electrons will typically involve energies on the scale of electron volts, which are defined to be

$$1\text{eV} = 1.602 \times 10^{-19}\text{joules} \tag{72}$$

which you may coincidentally recognize to be similar to the charge of an electron, which is $q = 1.602 \times 10^{-19}$ Coulombs (though its not really a coincidence).

Let's imagine we have a battery, and to make things really simple, let's make this battery a really stupid battery. We'll model it to be in one-dimension, with two barriers on each side confining in a whole bunch of electrons. Let's say that there are 100,000,000 electrons in the battery, each electron having an energy of $E = 1eV$. Let's also say that these barriers are pretty tall, maybe around 10000 times the energy of the electrons, or $V_0 = 10000eV$, and that they have a thickness of around one micrometer, which is $1 \times 10^{-6}$ meters. This is a bit of a simplified picture, but oh well, we just want to get a sense of the numbers. I've drawn it out in figure 26.

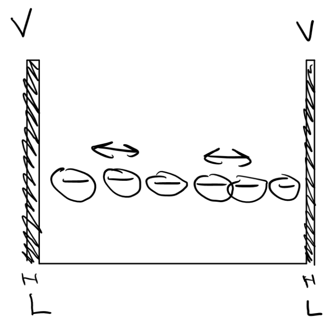

Figure 26: A cartoon of the toy model we have created for the battery.

You may be a bit confused on how to treat this new situation. We now have two walls and a bunch of different electrons. Are the electrons not going to bounce off of one another? Are they not going to repel each other? Aren't two walls going to change things? Indeed they will change things, but let's just get a rough sense of what's going on. We'll assume that the distance between the walls is much much larger than the distance an electron can travel, so we just have to solve for the tunneling probability for a single wall. We will also assume the electrons don't interact at all, so we can just solve for the tunneling probability for a single electron and generalize that same probability to all the electrons.

Ok, let's lay out the numbers. $E = 1eV$, $V_0 = 10000eV$, $m = 9.11 \times 10^{-31}kg$, $\hbar = 1.05 \times 10^{-34}Js$, $L = 1\mu m$ ($\mu m$ is the symbol for micrometer). Electron volts are a bit different from the usual SI units we usually work with, but they are a bit more convenient to work with when we are dealing with energy scales like these. Let's convert all of our other values to ones that are consistent with electron volts. Doing so gives us the following numbers

$$E = 1eV \tag{73}$$
$$V_0 = 10000eV \tag{74}$$
$$m = 511 \times 10^3 eV/c^2 \tag{75}$$
$$\hbar = 1.973 \times 10^5 eV\mu m/c \tag{76}$$
$$L = 1\mu m. \tag{77}$$

We just need to plug these into the equation for tunneling, which is

$$P_{tunneling} = 16\frac{E}{V}(1 - \frac{E}{V})e^{-2jL}. \tag{78}$$

Let's first calculate $j$:

$$j = \sqrt{-\frac{2m}{\hbar^2}(E - V)} \tag{79}$$
$$= \sqrt{\left(-\frac{2(511 \times 10^3)eV/c^2}{1.973 \times 10^5 eV\mu m/c}\right)(1eV - 10000eV))} \tag{80}$$
$$= 227.58\frac{1}{\mu m}. \tag{81}$$

Things are already looking messy... This is why its nice to just things to be equal to 1 sometimes. Now let's compute the full tunneling probability.

$$P_{tunneling} = 16\frac{E}{V}(1 - \frac{E}{V})e^{-2jL} \tag{82}$$
$$= 16\left(\frac{1eV}{10000eV}\right)\left(1 - \frac{1eV}{10000eV}\right) \tag{83}$$
$$\times \exp(-2(227.58\frac{1}{\mu m})(1\mu m)) \tag{84}$$
$$= 3.37 \times 10^{-201}. \tag{85}$$

Hmmmm.... Ok... So it looks like the tunneling probability as a percentage, is $3.37 \times 10^{-203}\%$. That's... pretty tiny, but it is nonzero, so its not impossible. If there are one hundred million electrons, then there is approximately a $3.37 \times 10^{-193}\%$ chance of some electron making it out. You ever find some really old batteries that have supposedly never been used before but for some reason don't hold any charge? Could it be that the electrons inside the battery have all tunneled out? With probabilities like that, quantum tunneling is probably not the reason. (The actual reason has to do with some chemistry thing.)

### 5.1.2 Example: Falling Through a Floor

Let's treat a fun but slightly stupid example. Everything is quantum right? Our bodies are comprised of protons and neutrons and electrons, all of which have quantum effects. What is there probability that all the particles in our body simultaneously tunnel through the floor, and we somehow fall through the floor?

Once again, we will make a lot of simplifications and approximations. Let's imagine that the "floor" is actually a barrier with a thickness of 1 meter, and that it has a barrier strength that is quite high, $10,000$ joules. Note that I am working with the usual SI units again as we are dealing with energy scales and length scales that we encounter in our day to day lives.

Instead of modeling our bodies as complex mixtures of different particles all arranged in an irregular shape, we'll just pretend we are comprised of a rectangle of non-interacting electrons, all with energy $1eV = 1.6 \times 10^{-19}J$. Let's say you are a rectangle of around 1.7 meters in height by 0.5 meters in width, making an area of 0.85 $m^2$. The classical electron radius is $r = 2.82 \times 10^{-15}m$, so the area of a single electron is $\pi r^2 = 2.50 \times 10^{-29}m^2$. So our entire body will comprise of $0.85/(2.50 \times 10^{-29}) = 3.4 \times 10^{-34}$ electrons. Let's list our values:

$$L = 1m \tag{86}$$
$$V_0 = 10,000J \tag{87}$$
$$E = 1.6 \times 10^{-19}J \tag{88}$$
$$\hbar = 1.05 \times 10^{-34}Js \tag{89}$$
$$m = 9.11 \times 10^{-31}kg \tag{90}$$

We'll return to our tunneling equation to try and figure out how likely it is we fall right through the floor.

First, we'll compute $j$:

$$j = \sqrt{-\frac{2m}{\hbar^2}(E - V)} \tag{91}$$
$$= \sqrt{-\frac{2(9.11 \times 10^{-31}kg)}{(1.05 \times 10^{-34}Js)^2}}(1.6 \times 10^{-19}J - 10,000J) \tag{92}$$
$$= 13172.84\frac{1}{m}. \tag{93}$$

The tunneling probability is then

$$P_{tunnel} = 16\frac{E}{V}(1 - \frac{E}{V})e^{-2jL} \tag{94}$$
$$= 16\frac{1.6 \times 10^{-19}}{10000}\left(1 - \frac{1.6 \times 10^{-19}}{10000}\right)\exp\left(-2 \times 13172.84\frac{1}{m} \times 1m\right) \tag{95}$$
$$= \epsilon. \tag{96}$$

You may wonder why I just wrote $\epsilon$ here... The tunneling probability in this case is actually so small that it is not calculable on a computer, so I just wrote in $\epsilon$ as I could not give you an actual numerical answer... Furthermore, this is only for a single electron. To calculate the probability for us to completely fall through the floor (meaning all of the electrons successfully tunnel through), we could have to calculate the probability all $3.4 \times 10^{28}$ electrons make it through, meaning the full probability would be

$$P_{\text{fall through the floor}} = \epsilon^{3.4\times10^{28}} \tag{97}$$

which a ton of absurdly small numbers multiplied together an absurd amount of times, yielding an even more absurdly small probability, but... not impossible I guess.

## 5.2 Applications of Tunneling: Microscopy

So we've done some funny sample calculations with quantum tunneling, but let us now consider an actual useful application of the quantum tunneling effect: Scanning tunneling microscopy.

In the earlier section, we saw that it actually was pretty likely for an electron to tunnel through if the barrier was thin. What could we apply this for? Well let's just come up with the most obvious thing: Imaging a really really thin surface (scanning tunneling microscopes can image surfaces as small as 0.1 nm, or $10^{-10}$ m thick!)

Regular microscopes are used to view really tiny things through clever applications of light, mirrors, and lenses. However, there is a fundamental limit on how small of an object can be imaged in such devices, and so we need to apply our knowledge of physics in clever ways to beat these limits.

Let's say we have a sheet of unknown material we want to image. This unknown material is lumpy and uneven, but is maybe at most a couple of atoms thick. To give you a sense of scale, a single carbon atom is around $0.0914nm = 9.14\times10^{-11}m$, meaning if we stacked 84666667 carbon atoms end to end, it would be about as long as a single banana. We can't really image such an object with a standard light-based microscope, so to map out its structure, we turn to quantum tunneling.

Recall that tunneling probability is quite sensitive to the thickness of the barrier. So, if we take a very fine electron gun and shoot electrons at a fixed rate at the surface we want to analyze, then we can figure out how thick the material is by checking how many electrons tunnel through to the other side (this is effectively a measure of the current). Thicker parts of the surface will end up by a lower current (less electrons can tunnel through the thicker potential barrier), while thinner parts will have a much higher current (more electrons tunnel through the thinner potential barrier). By very finely and systematically tracing the electron gun across every nook and cranny of the material and examining the current measured on the other side, we can map out the full structure of the surface profile!

The video "Quantum tunneling and how a scanning tunneling microscope works" by Alexandra Hopkins is a pretty good illustration of the process.

Of course, I could not end this section without some cool scanning tunneling microscopy images, shown in figure 27 and 28.

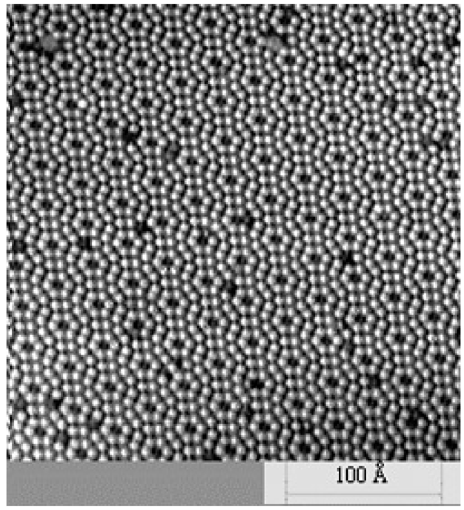


Figure 27: STM picture of a single layer of silicon. $100\mathring{A} = 10^{-8}m$. Photo taken from NREL,

# 6 Conclusion and Outlooks

So dear reader, you have made it (or skipped) to the end of these notes. You may not quite fully understand what a lot of these notes were talking about, but that's ok. I only hope the material was interesting and you are inspired to read and learn more about quantum theory! Certainly, things can be challenging as there is a bunch of mathematical formalism involved and oftentimes the physics is not as intuitive as one would like it to be. However, don't be discouraged if you struggled here, for I certainly didn't do the greatest job at explaining some of these things. Perhaps you left with more questions then when you came, and that's also ok! Let that be motivation for you to continue learning more physics and delving into this realm.

I have barely scratched the surface on the foundations of quantum theory, nor have I treated them with much rigor at all. These notes were just meant to be an appetizer, for there are many more interesting phenomena, such as entanglement, superpositions, no-cloning, teleportation, and much much more that I have not touched on at all. Furthermore, these concepts are not just weird science-fiction mumbo-jumbo, they are real-world, testable events that are happening all around us everyday! The world is indeed quantum, and we are currently in an extremely exciting time where we are beginning to be able to harness some of the power of quantum phenomena and also study it much more deeply. We are capable of holding single atoms in laser light and re-arranging them as we please, can prepare all sorts of interesting quantum states such as Schrodinger Cat states and GHZ states, and teleport quantum bits from the ground to satellites in space. Our engineering abilities have also advanced to such a degree that

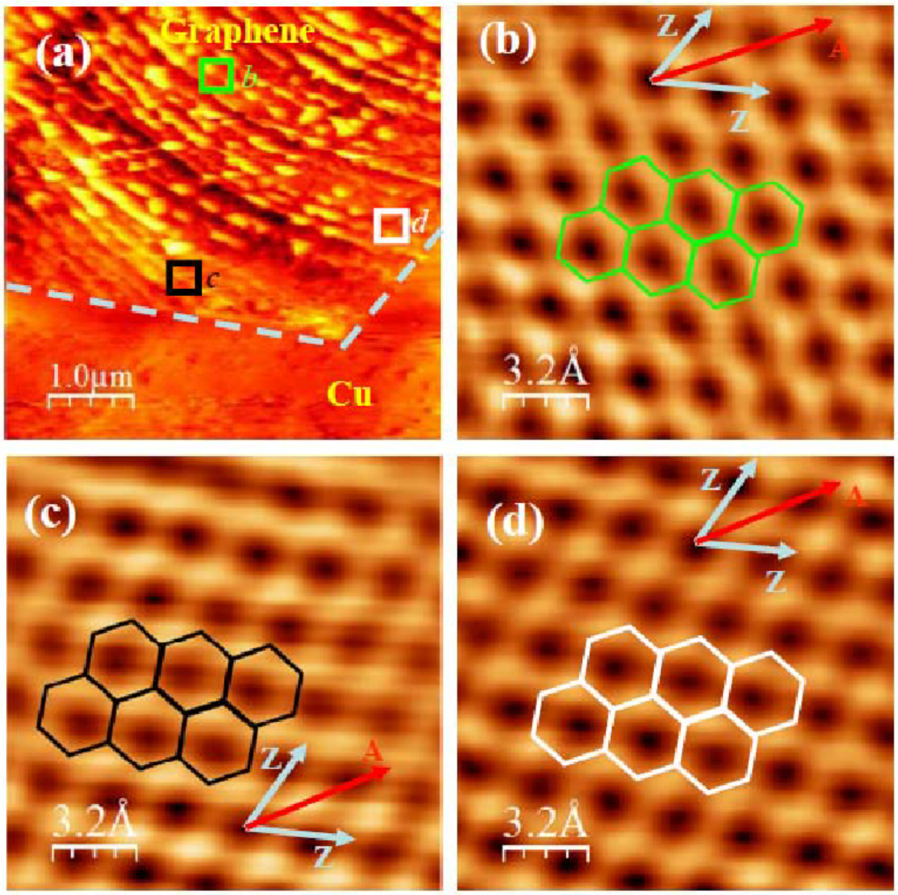


Figure 28: STM pictures of Graphene. Graphene is a layer of atoms arranged in a hexagonal lattice. (a) is taken near the corner of a single graphene grain, (b)-(d) are images taken of the surface. Note that the scale is in Angstroms, where 1 Angstrom is $10^{-10}$ meters. Tiny stuff! Figure taken from Yu. et al. (citation needed).

the devices we build (think of all of the tiny tiny transistors that power your computer) are experiencing quantum effects that must be aptly accounted for. Quantum is more relevant (and well-known in pop-culture) than ever!

Regardless of what you choose to continue in, you can't go wrong with learning more about quantum physics. Whether it be for a future career in research or in industry, or just to be a smart-ass to your friends, there are fascinating mathematics and experiments in quantum mechanics that are worth your time. Even if you do not go on to be a physicist or major in physics, I encourage you to at least take some more physics classes in college or read some books on similar topics!

# 7 Glossary of Relevant Experiments

Here I will list some experiments that demonstrate some of the quantum effects we have discussed in these notes. One of my biggest issues when I first began learning quantum mechanics was the disconnect between the theory and the math and the physical "how do we perform experiments and measure these quantities" parts of things. Hopefully this section will give you a better understanding of how experiments with quantum physics are done and how the theory is verified.

## 7.1 Measurements of atomic spectra

In the main section, we discussed how the emission spectrum of atoms are actually discrete. Here, I will discuss some of the methods for actually measuring the emission spectra of different atoms.

Some of the first observations of emission spectra were done in a surprisingly simple manner. By literally burning salts (for example NaCl, table salt) one could observe discrete colors fluoresce. In this case, the flames are used to excite the chemicals, which then in-turn lead to this emission.

In a more modern laboratory setting, one can probe the spectra of atoms a bit more precisely. We can now build cells containing fairly pure samples of certain atomic species, for example Rubidium. We can shine a laser beam through the atom and scan the frequency of the laser over a range of different frequencies and examine the output of the laser on a photodiode, which is a device that measures the intensity of the beam. You will then see dips in the intensity of the laser beam at the emission frequencies of the atom, as the laser light (at those specific frequencies) is being absorbed by the atoms instead of transmitting through to the detector.

The measurement of spectra is such a ubiquitous and important process that a lot of companies sell spectroscopy packages that enable easy measurements of the spectrum of different samples. One such product is shown in figure 29.


Figure 29: A mass-produced spectrometer, made by Vernier.

## 7.2 Measurement of the time a particle takes to tunnel through a barrier

This one is from a more modern experiment from 2020 by Ramos et al. Its a little hard to understand with what we have learnt so far, so I will do my best to simplify it.

Ramos et al. want to analyze how long it takes for a particle to actually tunnel through a barrier. This is a great question to ask! Here we have a particle making its way through a supposedly impassable barrier, so it is a natural question to wonder how long it will take for the particle to get through.

The particles in question are a Rubidium atoms, and the tunneling barrier is 1.3 micrometers thick. You may be wondering how you can time such a tiny and difficult to observe process, and this is where the cleverness of the experimentalists come to play! They use the precessions of the spin of the atoms in a magnetic field, a special quantum mechanical property of these atoms, which is discussed a bit in 8.3.

Before I fully describe the experiment, I will elaborate on this method of measuring time, known as a Larmor clock. You can imagine the spin of the atoms to be akin to the hand of a clock (although it really is nothing like that at all). It can be set to be in a certain direction, much like we can set the hand on our clock to a specific time. Let's say we set the hand on our clock to 1 o'clock. When a magnetic field put along a specific direction, say 6 o'clock, it can cause the spin of the atom to precess, meaning the hand will start to tick, rotating from one o'clock to two o'clock until it eventually aligns with the magnetic field at 6 o'clock.

Why might this be useful? Well, we can actually use our understanding of physics to derive a specific rate of precession for the clock hand. So if we set the spin of our atom to a specific "time" and let it tunnel through a barrier with a magnetic field applied in a pre-determined direction, we can then analyze the spin of the atom after it has tunneled through the barrier. From what "time" the spin now points to, we can infer how long the particle was in the barrier. This sort of set-up is shown in figure 30.

Using a variety of atomic physics methods, such as laser cooling and dipole traps, the

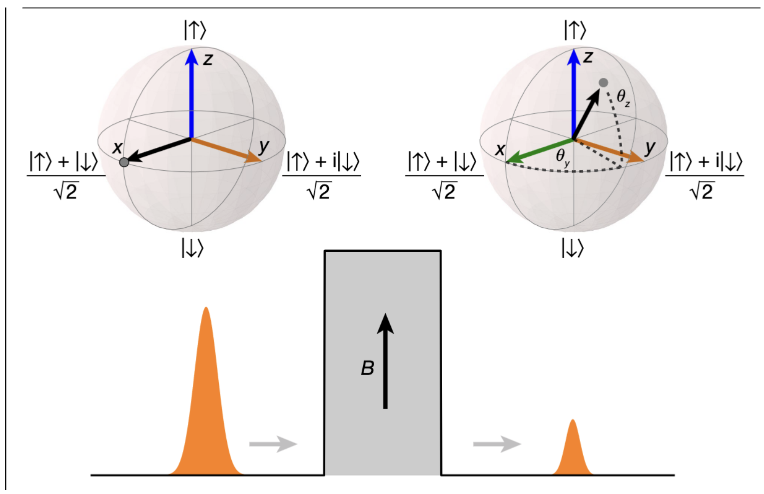


Figure 30: Example of a Larmor clock being applied to measuring the tunneling time. The globes in the top of the figure can be thought of as the clock face (don't worry about all the strange symbols). Initially, the researchers point the atom's spin clock hand along the x axis. The atom begins to tunnel through the barrier, which has a magnetic field in it. The magnetic field causes the spin clock hand to precess, until it ends up spinning an angle $\theta_y$ along the y-axis and an angle $\theta_z$ along the z axis. After the atom exits the barrier, the clock hands stop moving. From the angle, the time the particle spent in the variable can then be derived.

researchers prepare a large cloud of Rubidium particles in a specific quantum state. They use a set of lasers to generate a sort of pseudo-magnetic field that enables them to set the spin of the atoms to a specific "time" (which in this case is along the x-axis). They also use another magnetic field to push the cloud of atoms towards the barrier, where the tunneling process will then begin. The particles that successfully tunnel through the barrier then have their spin measured, from which the time they spent in the particle can then be inferred.

## 7.3 Tracing the Collapse of a Wavefunction

Earlier I discussed that a projective measurement causes the wavefunction to collapse to a spiky distribution known as an eigenstate. But how does the wavefunction go from a wide and featured distribution to a spike? Does it twist around, gradually shrink, before finally becoming sharp, or does it suddenly just shrink into a point? What is the trajectory of this collapse? I use my imagination and draw out some possibilities in figure 31.

So the projective measurement that I described is a very abrupt and "strong" measurement, which very quickly collapses the wave-function. In the paper by Murch et al., they perform a series of weak measurements which very slowly move the wave-function towards its final collapsed state. Essentially, performing a bunch of weak measurements eventually gets you to the same state as if you performed a projective measurement. By

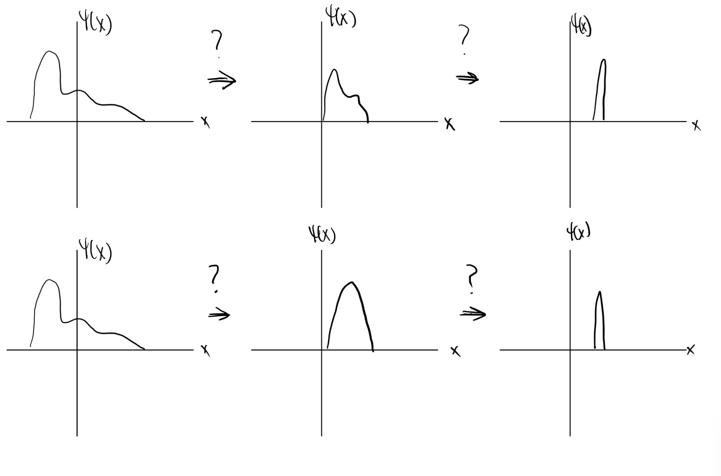


Figure 31: My random imaginings of how the wave-function collapses when projective measurements are made. Not to be taken seriously.

checking the state of the particle after weak measurements, they can essentially watch as the wave-function gradually makes its way towards collapse, giving us insight into the dynamics of measurement-based collapse.

The quantum particle that Murch et al. use is something known as a superconducting transmon. Its actually not an atom or anything "natural", its just a collection of special wires and components that mimic atomic properties. These types of architectures are pretty useful as they can be easily fabricated on chips and can easily be integrated with other devices.

Anyhow, the transmon is placed into a box, and microwaves are sent into the box. The microwaves weakly interact with the transmon (they don't talk to each other all that much) before leaving the box and arrive at a detector. From those microwaves, the researchers can glean some information about the quantum state of the transmon in the box and perform a weak measurement. The weak measurement slightly affects the quantum state of the transmon while other methods are used to figure out the state of the system after the measurement. This is done repeatedly to construct the trajectory of the quantum state over the course of successive measurements. A slightly confusing picture of the process is shown in figure 32.

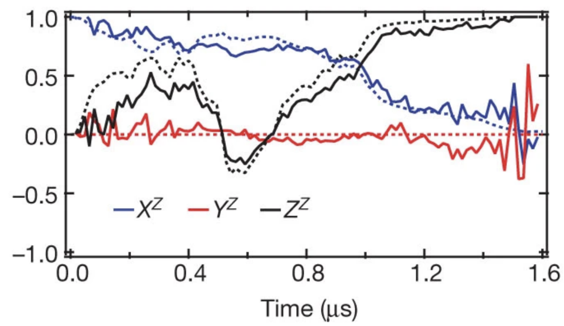


Figure 32: Units of the y-axis are in volts, which is the signal the researchers use to deduce information about the quantum state. Let's focus on the black line. You can see as the measurements are performed, the trajectory of the state almost wanders around randomly, before approaching 1, which corresponds to a collapse.

# 8 Glossary of Quantum and Classical Discrepancies

In this section, I will touch on some phenomena that demonstrate how quantum mechanics is truly different from classical mechanics. I will not explain any of these concepts too thoroughly, as some of them are quite advanced, and I will not really give you too much exposition on new notation or math concepts. They are just meant to demonstrate to you that quantum phenomena are indeed distinct from their classical counterparts.

## 8.1 Superposition

Whereas a classical bit (like those that comprise of our data) can be either a binary 1 OR 0, a quantum bit, or qubit, can be a 1, a 0, or in something known as a superposition of 1 and 0. A qubit in a superposition of 1 and 0 can be written, in a special notation known as Bra-Ket notation, to be

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,. \tag{98}$$

When the state is measured there is some probability $a^2$ that the qubit is in state 0, and a probability $b^2$ that the state is found in state 1. This is one of the fundamental principles quantum computing is built on. Whereas a single classical bit can either be 0 or 1, a quantum bit can be $|0\rangle$, $|1\rangle$, or $a\,|0\rangle + b\,|1\rangle$.

## 8.2 Entanglement

In quantum mechanics, we can generate entangled particles. Entangled particles are particles with quantum states that cannot be independently described. For example, if we have two entangled particles, it is mathematically impossible to describe them individually: the two particles must be treated as an inseparable hole. Even operations that are done to a single one of the entangled particles will affect the state of both of them.

Earlier, we talked about how projective measurement alters a the wavefunction of a quantum particle. Let's apply that to entanglement and see what happens. Let's say we have two entangled quantum particles and we give them to Schrodinger and his Cat. Schrodinger and his Cat then travel to very different countries. Let's say Schrodinger goes to Antartica and his cat goes to Africa. After a long while, Schrodinger becomes bored and measures his particle. Instantaneously, his cat's particle is also altered! This effect is the basis for protocols like quantum teleportation, where quantum information can be sent great distances at high speeds.

## 8.3 Spin

Spin is an intrinsic property of quantum particles that is always difficult to wrap your head around the first time you learn about it. Spin is a form of angular momentum carried by fundamental particles like electrons and protons. It doesn't really have that much of a classical counterpart. We can rotate the spins of particles with things like magnetic fields, and spins are an extremely important property that give rise to different classes of subatomic particles.

## 8.4 Wave-Particle Duality

You may have heard this one before. Quantum particles can actually either act like particles or act like waves depending on the situation they are in. The famous double slit experiment that you may have heard of is evidence of this.

## 8.5 Wigner Functions

A Wigner function describes the probability distribution of a quantum state in something known as phase space. For classical systems, you would find the usual positive-valued probability distributions you are accustomed to, but for quantum systems, the Wigner function can actually yield negative values. This has no classical analog. An example of a Wigner function with a negative-valued region is shown in figure **??**. Negative probabilities? Super weird! What does it mean? Don't ask. Quantum stuff I guess.